# Neutron channelling in a magnetic tube trap

V.G. Baryshevsky

*Research Institute for Nuclear Problems, 11 Bobryiskaya str., 220050, Minsk, Belarus*

Quantum effects at motion of ultracold neutrons in a magnetic trap are actively studied now. In particular, "magnetic tube" traps [1] are used for investigation of such effects. According to [1], the periodic set of magnetic traps can be developed. Traps in the form of "magnetic tube" are used for focusing of atomic beams and described in details in [2] on the basis of the classic motion equations for a particle, possessing the magnetic moment, in a magnetic field. But quantum effects, which appear at neutron motion in the magnetic trap, can not be described by classic motion equations. The typical radius of the magnetic trap, which is being designed, is $\rho \sim 10^{-4}$ cm [1]. Thus, due to large wavelength ($\lambda \sim 10^{-5} \div 10^{-6}$ cm), motion of the ultracold neutron in the magnetic trap in the direction, transversal to the axis of the magnetic tube, appears quantized.

Quantum effects were studied for neutron motion in a gravitational trap [3]. Neutron motion in such a trap is one-dimensional.

But in contrast to [3], transversal motion of neutrons in the "magnetic tube" trap with respect to it axes is two-dimensional. As noted below, neutron motion in such a trap is determined by spin-independent effective potential energy, which is proportional to the magnetic field strength square.

Thus, let us consider motion of the ultracold neutrons in the tube-shaped magnetic trap. The Schrödinger equation, which describes the ultracold neutron in such a trap, is as follows:

$$\left\{-\frac{\hbar^2}{2M}\Delta_r - \mu\vec{\sigma}\vec{B}(\vec{r})\right\}\Psi(\vec{r}) = E\Psi(\vec{r}) \tag{1}$$

where $M$ is the neutron mass, $\mu$ is the neutron magnetic moment, $\vec{r}$ is the coordinate of the neutron, $\vec{B}$ is the magnetic induction in the point $\vec{r}$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.

Suppose the axes $z$ to be directed along the tube axes. Neutron motion along the tube axes is quasi-classic. Tube length (dimension in direction $z$) significantly exceeds its radius. Therefore, $\vec{B}(\vec{r})$ inside the tube can be considered as independent on $z$, i.e $\vec{B}(\vec{r}) = \vec{B}(x, y) = \vec{B}(\rho, \varphi)$, where $\rho$ and $\varphi$ are the radius and azimuth angle in the cylindrical coordinate frame.

The equation (1) in the cylindrical coordinate frame looks like as follows:

$$\left\{-\frac{\hbar^2}{2M}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right]-\mu\vec{\sigma}\vec{B}(\rho,\varphi)\right\}\Psi = E\Psi \quad (2)$$

Particle motion along $z$ is free and is described by the quantum number $k_z$. Therefore, the wave function of a neutron in the tube can be expressed as:

$$\Psi(\vec{r}) = \Phi(\rho)e^{ik_z z} = \sum_{m=-\infty}^{\infty}\Phi_m(\rho)e^{im\varphi}e^{ik_z z} \quad (3)$$

Therefore, to find the spectrum of neutron states in the tube the following equation should be solved:

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}+\varkappa^2\right]\Phi_m+\frac{2M\mu}{\hbar^2}\sum_{m'}\vec{\sigma}\vec{B}_{mm'}(\rho)\Phi_{m'}(\rho) = 0 \quad (4)$$

where $\varkappa^2 = k^2 - k_z^2$, $k^2 = \frac{2ME}{\hbar^2}$, $\vec{B}_{mm'}(\rho) = \frac{1}{2\pi}\int e^{i(m'-m)\varphi}\vec{B}(\rho,\varphi)d\varphi$

Let us find the spectrum of neutron states in the trap in the first order of perturbation theory. In this case the diagonal matrix element of the interaction energy, i.e. the average value of the magnetic field strength $\vec{B}$ in the trap, determines the correction, which defines spectrum. But the averaged value of $\vec{B}$ in such a trap is zero. Therefore, the nonzero contribution to the spectrum of the neutron energy levels in the tube trap appears in the second order of the perturbation theory.

For further analysis it is convenient to transform differential equation (2) to the homogeneous integral equation:

$$\Psi(\vec{r}) = \int G_0(\vec{r}, \vec{r}')\hat{V}(\vec{r}')\Psi\vec{r}' d^3r' \tag{5}$$

where $G_0(\vec{r}, \vec{r}') = -\frac{M}{2\pi\hbar^2}\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$ is the Green function of the equation (2) and the potential energy $\hat{V} = -\mu\vec{\sigma}\vec{B}(\vec{r})$. Substitution of the expression for $\Psi(\vec{r})$ (5) to the right part of equation (5) gives:

$$\Psi(\vec{r}) = \int G_0(\vec{r}, \vec{r}')\hat{V}(\vec{r}')G_0(\vec{r}', \vec{r}'')\hat{V}(\vec{r}'')\Psi(\vec{r}'')d^3r'd^3r'' \tag{6}$$

This equation is equivalent to the following expression

$$\left\{-\frac{\hbar^2}{2M}\Delta_r - E\right\}\Psi(\vec{r}) + \int \hat{V}(\vec{r})G_0(\vec{r}, \vec{r}')\hat{V}(\vec{r}')\Psi(\vec{r}')d^3r' = 0 \tag{7}$$

Motion of the neutron along $z$ is quasiclassic, therefore the quasiclassic expression for the Green function can be used:

$$G_0(\vec{r}, \vec{r}') = -i\frac{M}{\hbar^2 k}\delta(\vec{\rho}' - \vec{\rho})\theta(z' - z)e^{ik(z'-z)} \tag{8}$$

where $\theta(z' - z)$ is the Heaviside unit function ($\theta(z' - z) = 1$, when $z' > z$ and $\theta(z' - z) = 0$, when $z' < z$).

With the help of (8) the expression (7) converts to

$$\left\{-\frac{\hbar^2}{2M}\Delta_r - E\right\}\Psi(\vec{r}) - i\frac{M}{\hbar^2 k}\hat{V}(\vec{\rho})\hat{V}(\vec{\rho})\int\limits_z^\infty e^{-ikz}e^{ikz'}\Psi(\vec{\rho},z')dz' = 0 \quad (9)$$

It should be mentioned that $\hat{V}(\vec{\rho})\hat{V}(\vec{\rho}) = \mu^2(\vec{\sigma}\vec{B})(\vec{\sigma}\vec{B}) = \mu^2\vec{B}^2(\vec{\rho})$.

With the help (3) the equation (9) can be expressed as

$$\left\{-\frac{\hbar^2}{2M}\Delta_\rho - \frac{\hbar^2\varkappa^2}{2M}\right\}\Phi(\vec{\rho}) + V_{eff}(\vec{\rho})\Phi(\vec{\rho}) = 0 \quad (10)$$

where

$$V_{eff}(\vec{\rho}) = \frac{M\mu^2 B^2(\vec{\rho})}{\hbar^2 k(k+k_z)} \quad (11)$$

is the effective potential energy.

According to (11) neutron motion in the magnetic tube trap, which is formed by the alternating magnetic field, is determined by the squared magnetic field strength and does not depend on the neutron spin direction.

The equation (10) allows to find the eigenfunctions and the spectrum of eigenvalues of neutron energy in the trap.

In the cylindrical coordinate frame the equation (10) reads as follows:

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{M^2}{\rho^2}+\varkappa^2\right)\Phi_m(\rho)-\frac{2M}{\hbar^2}\sum_{m'}\langle m|\hat{V}_{eff}(\vec{\rho})|m'\rangle\Phi_{m'}(\vec{\rho})=0(12)$$

here $\Phi(\vec{\rho})=\sum_m \Phi_m(\vec{\rho})e^{im\varphi}$, $\langle m|\hat{V}_{eff}(\vec{\rho})|m'\rangle=\frac{1}{2\pi}\int_0^{2\pi} V_{eff}(\vec{\rho},\varphi)e^{-i(m-m')\varphi}d\varphi$.

Let us keep in the sum in (12) only the term with $m'=m$, which describes the effective potential energy $V_{eff}(\vec{\rho},\varphi)$ averaged over the azimuth angle $\varphi$. (More detailed analysis of neutrons in the trap could require consideration of additional terms in the sum with $m'\neq m$.) Then (12) looks like as follows:

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{M^2}{\rho^2}+\varkappa^2\right)\Phi_m(\rho)-\frac{2M}{\hbar^2}\langle m|\hat{V}_{eff}(\vec{\rho})|m\rangle\Phi_m(\vec{\rho})=0(13)$$

According to [?] magnetic field in the tube trap linearly decreases to the trap center, i.e. $\vec{B}(\vec{\rho})\sim\rho$. Therefore, $B^2(\vec{\rho})\sim\rho^2$ and $\langle m|\hat{V}_{eff}(\vec{\rho})|m\rangle\sim\rho^2$. Therefore, equation (13) can be rewritten as:

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{M^2}{\rho^2}+\varkappa^2\right)\Phi_m(\rho)-A\rho^2\Phi_m(\vec{\rho})=0 \qquad (14)$$

where $A$ is determined by

$$A\rho^2=\frac{2M}{\hbar^2}\langle m|\hat{V}_{eff}(\vec{\rho})|m\rangle=\frac{2M\mu^2\langle m|B^2(\vec{\rho})|m\rangle}{\hbar^4k(k+k_z)} \qquad (15)$$

The eigenvalues and eigenfunctions of the obtained equation (14) are the those of the cylindrically symmetric oscillator [4]. Introducing new variable $\xi = \sqrt{A}\rho^2$ one can get from (14) the following equation:

$$\xi \frac{\partial^2 \Phi_m}{\partial \xi^2} + \frac{\partial \Phi}{\partial \xi} + (-\frac{\xi}{4} + \beta - \frac{m^2}{4\xi})\Phi_m = 0 \tag{16}$$

where $\beta = \frac{\varkappa^2}{4\sqrt{A}}$.

Solution of this equation is as follows [**?**]:

$$\Phi_m(\xi) = e^{-\frac{\xi}{2}} \xi^{\frac{|m|}{2}} W(\xi) \tag{17}$$

where $W(\xi)$ is the confluent hypergeometric function

$$W(\xi) = F\left\{-(\beta - \frac{|m|+1}{2}), |m|+1, \xi\right\}. \tag{18}$$

To make the wavefunction $\Phi_m$ finite everywhere, the parameter $\beta - \frac{|m|+1}{2}$ should be integer nonnegative number, i.e.

$$\beta - \frac{|m|+1}{2} = n_\rho, \text{ where } n_\rho \geq 0. \tag{19}$$

This condition (19) determined spectrum of the transversal motion for the neutron in the magnetic trap.

To avoid misunderstanding, I'd like to emphasize that I consider spectrum of the energy levels, which are lower then the height of the effective potential barrier.

Substitution $\beta = \frac{\varkappa^2}{4\sqrt{A}}$ in the explicit form to (19) provides to get:

$$\frac{\varkappa^2}{4\sqrt{A}} - \frac{|m|+1}{2} = n_\rho, \text{ where } n_\rho \geq 0, \tag{20}$$

therefore

$$E_{trans}(n_\rho, m) = \frac{\hbar^2 \varkappa^2(n_\rho, m)}{2M} = \frac{\hbar^2}{2M} 4\sqrt{A}(n_\rho + \frac{|m|+1}{2}) \tag{21}$$

Let us consider the particular example. The averaged field can be expressed as

$$\langle m|B^2(\vec{\rho})|m\rangle = \frac{\langle B^2_{max}\rangle}{\rho^2_{max}}\rho^2 \tag{22}$$

where $\rho_{max}$ is the radius of the magnetic tube, $B^2_{max}$ is the square averaged magnetic field at $\rho = \rho_{max}$. In this case the coefficient $A$ is

$$A = \frac{2M^2}{\hbar^2} \frac{\mu^2 \langle B^2_{max}\rangle}{k(k+k_z)\rho^2_{max}} \tag{23}$$

The frequency of the transverse oscillations of the neutron in the ground state in the magnetic tube

$$\Omega_{trans} = \frac{\hbar\sqrt{A}}{M} = \sqrt{2}\frac{\mu\sqrt{\langle B^2_{max}\rangle}}{\hbar\sqrt{k(k+k_z)}\rho_{max}} \tag{24}$$

For the ultracold neutrons with $\lambda \approx 10^{-6} \div 10^{-5}$ cm in the tube with $\rho_{max} \sim 10^{-4}$ cm the product $\sqrt{k(k+k_z)}\rho_{max} \approx k\rho_{max} \gg 1$. Therefore, for the magnetic field $\sqrt{\langle B_{max}^2\rangle} = 10^4$ gauss the frequency $\Omega_{trans} \approx 10^4 \div 10^5$ s$^{-1}$. Suppose that during the time $\tau$ the neutron moves with the speed $10^2$ cm/s in the tube of 1 cm length (time of neutron flight in the trap is about $\tau \sim 10^{-2}$ s), then the parameter $\Omega_{trans}\tau \gg 1$, i.e. neutron executes many oscillations in the trap.

Thus, neutron motion in the magnetic tube is described by the energy of the transversal motion $E_{trans}$ and the energy of longitudinal motion $E_{long} = \frac{\hbar^2 k_z^2}{2M}$. The sum of these energies is equal to the neutron energy before it flies into the trap $E = \frac{\hbar^2 k^2}{2M}$:

$$E = E_{trans} + E_{long}. \tag{25}$$

The equation (25) can be used to find $k_{\|}$ for the neutron in the certain state of transversal motion.

When the energy of transversal motion is comparable or exceeds the height of the effective potential barrier, the above expressions for the wavefunctions and $E_{trans}$ cannot be applied.

The typical angle of neutron incidence on the magnetic tube trap providing the energy of transversal motion to be less than the height of the effective potential barrier can be expressed as:

$$\vartheta_{cr} \sim \sqrt{\frac{V_{eff}}{E}} \approx \frac{\Omega_L}{E/\hbar}. \tag{26}$$

where $\Omega_L$ is the frequency of Larmor precession of neutron spin in the field $\sqrt{\langle B_{max}^2 \rangle}$. The expression (26) was obtained considering $k \approx k_z$ (because $k_z$ converges to $k$ with energy growth).

According to (26) the neutron moving with the speed 1 m/s in the trap with the maximal field $B \approx 10^4$ gauss can be captured to the trap with the potential $V_{eff}$ at the angle $\vartheta < \vartheta_{cr} \approx 1$ rad. When neutron speed is $v = 10$ m/s, the angle $\vartheta_{cr} \approx 10^{-2}$ rad. Energy growth results in reducing of $\vartheta_{cr} \sim \frac{1}{E} \sim \frac{1}{v^2}$

Neutron passing through the magnetic tube trap is similar to the process of axial channelling of a charged particle in a crystal. Study of the angular distribution for neutrons passing through the trap provides to find spectrum of the neutron levels in the trap.

Actually, experiments for neutrons(atoms) passing through the magnetic tube trap can be considered as experiments studying neutron (atom) channelling in such a trap. When the traps are periodically distributed in the plane $(x, y)$ the analogy becomes ever more evident.

It should be also mentioned that bent magnetic tube traps can be used for focusing wide beams of ultra cold neutrons (similar focusing of particles channelled in a crystal).

## 1 Acknowledgements

I wish to thank V.F. Ezhov, who initiated topic discussion.

## References

[1] V.F. Ezhov, B.A. Bazanov, P. Geltenbort et.al. JTP Lett., **27**, p.1055 (2001).

[2] Norman F. Ramsey "Molecular Beams" The Claredon Press, 1956.

[3] V.V. Nesvizhevsky, Quantum states of neutrons in a gravitational field and the interaction neutrons with nanoparticles, Joint scientific session of the Physical Sciences Department of the Russian Academy of Sciences and the Joint Physical Society of the Russian Federation (25 September 2002) no.1,102.

[4] L. D. Landau and E. M. Lifshitz, *Quantum mechanics*, Pergamon Press, London, 3rd edition, (1977).