Equation of spin motion in storage rings in a cylindrical coordinate system

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Abstract

The exact equation of spin motion in a cylindrical coordinate system with allowance for electric dipole moments of particles has been derived. This equation is convenient for analytical calculations of spin dynamics in circular storage rings when the configuration of main fields is simple enough. The generalized formula for the influence of a vertical betatron oscillation on the angular velocity of spin rotation has been found. This formula agrees with the previously obtained result and contains an additional oscillatory term that can be used for fitting. The relative importance of terms in the equation of spin motion is discussed.

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I. INTRODUCTION

New experiments for measurement of the muon anomalous magnetic moment (AMM) [1, 2] and the electric dipole moments (EDMs) of fundamental particles [3, 4] in storage rings need an extremely high accuracy of measurement of the spin precession in storage rings. To distinguish the spin precession induced by the EDMs of particles and other minor effects, the spin motion has to be mathematically described in a convenient manner. The goal of this paper is to derive the equation in cylindrical coordinates which allows an exact analytical description of spin dynamics in storage rings. We do not confine ourselves to specific configurations of electric and magnetic fields. In particular, electric or magnetic focusing fields and an accelerating electric field can be used. The influence of EDMs on the spin dynamics is taken into account. We consider the problem of averaging the angular velocity of spin rotation and calculate the generalized formula for the influence of vertical betatron oscillation upon this quantity.

There are many algorithms of computer calculations based on different coordinate systems (e.g., on Frenet-Serret coordinates [5]). These algorithms cover any problem of beam and spin dynamics in storage rings. However, an analytical solution of the problem can be necessary for several high-precision experiments. We mean the g$-$2 [1, 2] and EDM [3, 4] experiments, those precision is extremely high. These experiments need a clear understanding of spin dynamics. The use of cylindrical coordinates for analytical calculations of spin dynamics can be very successful if the configuration of main fields is simple enough. For circular storage rings, it is quite natural to describe the spin motion just in cylindrical coordinates. For this reason, other formalisms are less convenient for the g$-$2 and EDM experiments. This problem will be considered properly later.

The relativistic system of units $\hbar = c = 1$ is used.

II. GENERAL EQUATIONS OF PARTICLE AND SPIN MOTION

The particle trajectory in a circular storage ring is approximately a circle. Coherent betatron oscillations (CBOs) of a beam as a whole in the horizontal and vertical planes affect the trajectories of individual particles. Synchrotron motion occurs whenever radio frequency (rf) cavities are used. The incoherent motion of particles can also take place. As a result, the particle trajectories are never closed. Field defects and misalignments of magnets, electric field plates and electrostatic quadrupoles cause particle trajectory distortions. These effects lead to a change in the magnetic field acting on the spin in the particle rest frame. As a result, the spin motion becomes complicated. For this reason, equations of spin motion correctly taking into account perturbations of the particle trajectory have only been obtained for some specific problems [6, 7, 8]. In Refs. [7, 8], the corrections to the spin rotation frequency due to the vertical and radial betatron oscillations have been calculated.

As a rule, the use of the one-particle approximation is sufficient to describe the particle and spin motion. In this approximation, coherent and incoherent betatron oscillations give similar effects. However, the CBOs are more important than the incoherent BOs because they cause a systematic shift in the spin precession frequency in the real experiments (see Refs. [1, 2]).

The particle motion is described by the Lorentz equation

$$\frac{d\boldsymbol{p}}{dt} = e\left(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}\right), \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c} = \frac{\boldsymbol{p}}{\gamma m}. \tag{1}$$

It is convenient to use the unit vector of momentum direction $\boldsymbol{N} = \boldsymbol{p}/p$ which defines the direction of particle motion. Since

$$\frac{d\boldsymbol{N}}{dt} = \frac{\dot{\boldsymbol{p}}}{p} - \frac{\boldsymbol{p}}{p^3}\left(\boldsymbol{p}\cdot\dot{\boldsymbol{p}}\right),$$

Eq. (1) takes the form

$$\frac{d\boldsymbol{N}}{dt} = \boldsymbol{\omega}\times\boldsymbol{N}, \quad \boldsymbol{\omega} = -\frac{e}{\gamma m}\left(\boldsymbol{B} - \frac{\boldsymbol{N}\times\boldsymbol{E}}{\beta}\right), \tag{2}$$

where $\boldsymbol{\omega}$ is the angular velocity of particle rotation.

The spin motion is determined by the Thomas-Bargmann-Michel-Telegdi (T-BMT) equation

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\Omega}_{T-BMT}\times\boldsymbol{s},$$
$$\boldsymbol{\Omega}_{T-BMT} = -\frac{e}{2m}\left\{\left(g-2+\frac{2}{\gamma}\right)\boldsymbol{B} - \frac{(g-2)\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B}) - \left(g-2+\frac{2}{\gamma+1}\right)(\boldsymbol{\beta}\times\boldsymbol{E})\right\}, \tag{3}$$

where $\boldsymbol{s}$ is the spin vector. This equation was derived by Thomas [9] (also by Frenkel [10]) and, in a more general form, by Bargmann, Michel and Telegdi [11]. In this work, "spin" means an expectation value of a quantum mechanical spin operator.

The T-BMT equation describes the motion of spin in the rest frame of the particle, wherein the spin is described by a three-component vector. The longitudinal direction in this frame is mapped from the longitudinal direction in the laboratory frame.

A comparison of Eqs. (2) and (3) shows that the spin of the particle placed in a vertical magnetic field rotates in the horizontal plane with a frequency proportional to g−2. We use the term "g−2 precession" for any spin rotation in the horizontal plane, even in the presence of an electric field.

III. CORRECTIONS TO THE ANGULAR VELOCITY OF PARTICLE MOTION IN THE HORIZONTAL PLANE

The particle spin motion in storage rings is usually specified with respect to the particle trajectory. The use of the cylindrical coordinates considerably simplifies the analysis of g−2 precession and effects caused by the particle EDM. The particle rest frame is less convenient for taking into account the acceleration of the particle and the tilt of its orbit. When the configuration of the main fields is simple enough, the use of cylindrical coordinates simplifies the form of equations of particle and spin motion. However, the description of the spin motion in the cylindrical coordinate system is a difficult problem because the axes of this system are defined by the position of the particle being in oscillatory motion. The transformation of the T-BMT equation to the cylindrical coordinates should be performed with allowance for the oscillatory terms in the particle motion equation.

The vertical CBO and orbit distortions change the plane of particle motion. The (pseudo)vector of angular velocity becomes tilted. The instantaneous plane of particle motion does not coincide with the horizontal plane, and the instantaneous plane of rotation of vector $\boldsymbol{N}$ is not horizontal. The angle Φ between two positions of the rotating vector $\boldsymbol{N}$ in the tilted plane is not equal to the angle ϕ between two corresponding horizontal projections. Therefore, the vertical CBO and the orbit distortions change the instantaneous angular velocity of particle motion. This effect can be calculated.

We suppose the x- and y-axes are horizontal and the z-axis is vertical. Since we use the cylindrical coordinate system, it is convenient to direct the z-axis orthogonally to the plane of the unperturbed particle motion. We can define the angle of particle rotation in the xy-plane as the angle between two horizontal projections of the vector of momentum direction in two indicated positions, $\boldsymbol{N}_{||}$ and $\boldsymbol{N}_{||}'$. The infinitesimal angle of particle rotation in the xy-plane, $d\phi$, is given by

$$d\phi=\frac{(\boldsymbol{N}_{||}\times\boldsymbol{N}_{||}')\cdot\boldsymbol{e}_z}{|\boldsymbol{N}_{||}|\cdot|\boldsymbol{N}_{||}'|}=\frac{(\boldsymbol{N}_{||}\times d\boldsymbol{N}_{||})\cdot\boldsymbol{e}_z}{|\boldsymbol{N}_{||}|^2},$$

where $d\boldsymbol{N}_{||}=\boldsymbol{N}_{||}'-\boldsymbol{N}_{||}$ and $d\boldsymbol{N}_{||}$ is an infinitesimal vector. The sign $||$ means a horizontal projection for any vector. The quantity $d\boldsymbol{N}_{||}$ determines the deflection of the particle momentum direction in a time interval dt. The instantaneous angular velocity of particle rotation in the horizontal plane is equal to

$$\dot{\phi}\equiv\frac{d\phi}{dt}=\frac{(\boldsymbol{N}_{||}\times\dot{\boldsymbol{N}}_{||})\cdot\boldsymbol{e}_z}{|\boldsymbol{N}_{||}|^2}=\omega_z-o, \tag{4}$$

where

$$o=\frac{(\omega_xN_x+\omega_yN_y)N_z}{1-N_z^2}=\frac{(\omega_\rho N_\rho+\omega_\phi N_\phi)N_z}{1-N_z^2}. \tag{5}$$

Different components of vector $\boldsymbol{\omega}$ are defined by Eq. (2). Indexes ρ and ϕ mean projections onto the basis vectors $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$ of the cylindrical coordinate system.

Eqs. (4),(5) are exact. We can confirm the validity of these equations by the fact that they perfectly describe a constant tilt of the particle orbit and then evaluate the quantity o in experimental conditions when the tilt of the particle orbit oscillates.

If the particle orbit is perfectly horizontal, the momentum direction is defined by

$$N_x=-\sin{(\omega_0t+\varphi_0)},\quad N_y=\cos{(\omega_0t+\varphi_0)},\quad N_z=0,$$

where ω_0 is the cyclotron frequency and φ_0 is an arbitrary phase. If the normal to the tilted particle orbit is orthogonal to the y-axis and deflected from the z-axis by a constant angle θ, y-components of all vectors remain unchanged and the components of vectors $\boldsymbol{\omega}$ and $\boldsymbol{N}$ are equal to

$$\begin{gathered}\omega_x=\omega_0\sin\theta,\quad \omega_y=0,\quad \omega_z=\omega_0\cos\theta,\\ N_x=-\cos\theta\sin{(\omega_0t+\varphi_0)},\quad N_y=\cos{(\omega_0t+\varphi_0)},\quad N_z=\sin\theta\sin{(\omega_0t+\varphi_0)}.\end{gathered} \tag{6}$$

In this case, Eq. (4) takes the form

$$\dot{\phi}=\frac{\omega_0\cos\theta}{1-\sin^2\theta\sin^2{(\omega_0t+\varphi_0)}}. \tag{7}$$

As a result of integrating and averaging over time, we obtain $<\dot{\phi}>=\omega_0$. This result agrees with the evident fact that the average frequencies of particle motion in the tilted and horizontal planes are equal and therefore confirms the validity of Eqs. (4),(5).

To evaluate the quantity o in real experimental conditions, we may confine ourselves to taking into account only the particle trajectory perturbations caused by the radial and vertical CBOs. The synchrotron motion does not affect this quantity. By our estimate, it is sufficient to approximate the horizontal and vertical betatron motion by the simple forms

$$N_\rho = \frac{p_\rho}{p} = \rho_0 \sin(\omega_r t + \alpha),$$
$$N_z = \frac{p_z}{p} = \psi_0 \sin(\omega_v t + \delta),$$

where ρ_0 and ψ_0 are angular amplitudes, ω_r and ω_v are angular frequencies of the radial and vertical CBOs, respectively.

With allowance for orders of quantities

$$\omega_\rho \sim \psi_0\omega_v, \quad \omega_\phi \sim \rho_0\psi_0\omega_v, \quad N_\rho \sim \rho_0, \quad N_\phi \approx \pm 1, \quad N_z \sim \psi_0, \tag{8}$$

we obtain that the quantity o is of the third order in the angular amplitudes ρ_0 and ψ_0. Moreover, it oscillates and therefore averages to zero. If we take into account only the second-order terms in the angular amplitudes and the average particle orbit is not tilted, the quantity o is negligible. Approximately,

$$\dot{\phi} = \omega_z = -\frac{e}{\gamma m}\left(B_z - \frac{(\boldsymbol{N}\times\boldsymbol{E})_z}{\beta}\right). \tag{9}$$

IV. EQUATION OF SPIN MOTION IN CYLINDRICAL COORDINATES

To transform the general equation of spin motion (3) to cylindrical coordinates, we need to find the quantities

$$\frac{ds_\rho}{dt}, \quad \frac{ds_\phi}{dt}, \quad \text{and} \quad \frac{ds_z}{dt}.$$

It follows from the geometry of the problem that the horizontal axes $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$ rotate with the instantaneous angular velocity

$$\boldsymbol{\omega}' = \dot{\phi}\boldsymbol{e}_z.$$

It is easy to show by simple transformations that the spin motion with respect to the axes of the cylindrical coordinate system can be written in the form

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\omega}_a \times \boldsymbol{s}, \quad \boldsymbol{\omega}_a = \boldsymbol{\Omega} - \dot{\phi}\boldsymbol{e}_z. \tag{10}$$

In this equation, $\boldsymbol{\omega}_a$ is the the angular velocity of spin rotation in cylindrical coordinates. The corresponding angular velocity in Cartesian coordinates equals $\boldsymbol{\Omega}$. If the spin motion is described by the T-BMT equation, $\boldsymbol{\Omega}=\boldsymbol{\Omega}_{T-BMT}$. The difference between the quantities $\boldsymbol{\omega}_a$ and $\boldsymbol{\Omega}$ is caused by the rotation of the axes $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$.

The equation of spin motion with allowance for the EDM can be obtained by modifying the T-BMT equation. The EDM and AMM determine the real and imaginary parts of the same interaction Lagrangian [12, 13]. This Lagrangian is used to derive an extended Dirac equation. The contributions of the EDM and AMM to the Lagrangian are equal to [12]

$$\mathcal{L}_{AMM}=\frac{\mu'}{2}\sigma^{\mu\nu}F_{\mu\nu},\qquad \mathcal{L}_{EDM}=-i\frac{d}{2}\sigma^{\mu\nu}\gamma^5F_{\mu\nu},\quad \sigma^{\mu\nu}=\frac{i}{2}\left(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right),\tag{11}$$

where $\mu'=e(g-2)/(4m)$ is the AMM, d is the EDM, γ^μ ($\mu=0,1,2,3$) are the Dirac matrices, and $F_{\mu\nu}$ is the electromagnetic field tensor. The use of the explicit forms of $\mathcal{L}_{AMM}$ and $\mathcal{L}_{EDM}$ shows that the contribution of the EDM to the full Lagrangian can be obtained from the corresponding contribution of the AMM by the substitution

$$\boldsymbol{B}\rightarrow\boldsymbol{E},\quad \boldsymbol{E}\rightarrow-\boldsymbol{B},\quad \mu'\rightarrow d.\tag{12}$$

Taking into account the particle EDM leads to the following modification of the T-BMT equation [14, 15, 16]:

$$\begin{gathered}\frac{d\boldsymbol{s}}{dt}=\boldsymbol{\Omega}\times\boldsymbol{s},\quad \boldsymbol{\Omega}=\boldsymbol{\Omega}_{T-BMT}+\boldsymbol{\Omega}_{EDM},\\ \boldsymbol{\Omega}_{EDM}=-\frac{e\eta}{2m}\left(\boldsymbol{E}-\frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E})+\boldsymbol{\beta}\times\boldsymbol{B}\right),\end{gathered}\tag{13}$$

where $\boldsymbol{\Omega}_{T-BMT}$ is defined by Eq. (3) and $\eta=4dm/e$.

The EDM affects the particle motion if the electric field is not uniform. However, the correction to the equation of particle motion is negligible. If rf cavities are not used, we can also neglect the term $\frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E})$.

The transformation of Eq. (13) to cylindrical coordinates is given by Eq. (10), where

$$\begin{aligned}\boldsymbol{\omega}_a=-\frac{e}{m}\Biggl\{&a\boldsymbol{B}-\frac{a\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})\\ &+\left(\frac{1}{\gamma^2-1}-a\right)(\boldsymbol{\beta}\times\boldsymbol{E})+\frac{1}{\gamma}\left[\boldsymbol{B}_{\|}-\frac{1}{\beta^2}(\boldsymbol{\beta}\times\boldsymbol{E})_{\|}\right]\\ &+\frac{\eta}{2}\left(\boldsymbol{E}-\frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E})+\boldsymbol{\beta}\times\boldsymbol{B}\right)\Biggr\}+o\boldsymbol{e}_z,\quad a=\frac{g-2}{2}.\end{aligned}\tag{14}$$

This formula is exact, and $\boldsymbol{\omega}_a$ is the angular velocity of spin precession. Eqs. (10),(14) describe the spin motion in arbitrary storage rings with allowance for the EDM. The term

$o\boldsymbol{e}_z$ in formula (14) is negligible for the EDM and g−2 experiments. Measurements of the AMM in the g−2 experiment are performed at an energy such that $1/(\gamma^2-1)=a$, that is $\gamma=29.3$. In this case, the third term in Eq. (14) is equal to zero [1, 2].

After neglecting the small terms, the equation for the angular velocity of the g−2 precession with allowance for the EDM takes the form

$$\boldsymbol{\omega}_a = -\frac{e}{m}\Biggl\{a\boldsymbol{B} - \frac{a\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B}) + \left(\frac{1}{\gamma^2-1}-a\right)(\boldsymbol{\beta}\times\boldsymbol{E}) + \frac{1}{\gamma}\left[\boldsymbol{B}_{\|} - \frac{1}{\beta^2}(\boldsymbol{\beta}\times\boldsymbol{E})_{\|}\right] + \frac{\eta}{2}(\boldsymbol{E}+\boldsymbol{\beta}\times\boldsymbol{B})\Biggr\}. \tag{15}$$

Formulae (14),(15) are useful for analytical calculations of spin dynamics in cylindrical coordinates with allowance for field misalignments and beam oscillations.

V. AVERAGING OF THE ANGULAR VELOCITY OF SPIN ROTATION

The spin dynamics given by formula (14) is simple when the tilt of the spin out of the xy-plane accumulates from turn to turn. Such a behavior of the spin takes place in the planned EDM experiment. However, in the g−2 experiment one needs to determine the average and integral characteristics of the spin motion in the horizontal plane [1, 2]. The instantaneous angular velocity of spin rotation in the horizontal plane, $\dot\psi$, is characterized by the change of angle ψ determining the spin orientation in this plane. The quantity $\dot\psi$ can be found similarly to the corresponding quantity $\dot\phi$ describing the particle rotation (Section III). It is given by

$$\dot\psi \equiv \frac{d\psi}{dt} = \frac{(\boldsymbol{\xi}_{\|}\times\dot{\boldsymbol{\xi}}_{\|})\cdot\boldsymbol{e}_z}{|\boldsymbol{\xi}_{\|}|^2} = (\omega_a)_z - O, \tag{16}$$

where

$$O = \frac{\left[(\omega_a)_x\xi_x + (\omega_a)_y\xi_y\right]\xi_z}{1-\xi_z^2} = \frac{\left[(\omega_a)_\rho\xi_\rho + (\omega_a)_\phi\xi_\phi\right]\xi_z}{1-\xi_z^2} \tag{17}$$

and $\boldsymbol{\xi}=\boldsymbol{s}/s$.

As an example, we can derive the generalized formula for the average value of $\dot\psi$ affected by the vertical CBO. When the particle and spin motion is unperturbed, the angular velocity of spin rotation is equal to

$$\Omega_0 = \frac{|ea|}{m}B_z. \tag{18}$$

We suppose that $B_z > 0$.

The vertical CBO gives an important correction to the g−2 frequency (the pitch correction). The formula used currently for this correction has been calculated by Farley [7]. The result has been confirmed by Field and Fiorentini [8] and by computer simulations [17]. If we ignore the possible existence of the EDM, then Eq. (15) can be written in the form

$$\begin{gathered}\frac{d\boldsymbol{\xi}}{dt} = \{a_0 + a_3\cos\left[2(\omega_v t + \delta)\right]\}\left(\boldsymbol{e}_z \times \boldsymbol{\xi}\right) \\ +a_2\sin\left(\omega_v t + \delta\right)(\boldsymbol{e}_\phi \times \boldsymbol{\xi}) + a_1\cos\left(\omega_v t + \delta\right)(\boldsymbol{e}_\rho \times \boldsymbol{\xi}),\end{gathered} \tag{19}$$

where

$$\begin{gathered}a_0 = \lambda\Omega_0\left(1 - \frac{\gamma - 1}{2\gamma}\psi_0^2\right), \\ a_1 = \lambda f\omega_v\psi_0, \quad a_2 = -\Omega_0\frac{\gamma-1}{\gamma}\psi_0, \\ a_3 = \lambda\Omega_0\frac{\gamma-1}{2\gamma}\psi_0^2,\end{gathered}$$

and λ is equal to 1 and -1 for negative and positive muons, respectively. The factor f is given by

$$f = 1 + a\gamma - \frac{1+a}{\gamma} = 1 + a\beta^2\gamma - \frac{1}{\gamma}$$

and

$$f = 1 + a\gamma$$

for electric and magnetic focusing, respectively. In the muon g−2 experiment, $\psi_0 \sim 10^{-3}$. It can be shown that the effect of a_3 on the spin dynamics described by Eq. (19) is negligible.

If we substitute the zero-order approximation

$$\xi_\rho^{(0)} = \xi_\| \cos\left(a_0 t + \phi_0\right), \quad \xi_\phi^{(0)} = \xi_\| \sin\left(a_0 t + \phi_0\right), \quad \xi_z^{(0)} = \xi_\perp \tag{20}$$

into the right side of Eq. (19), the first-order approximation is given by

$$\begin{gathered}\xi_\rho = \xi_\|\cos\left(a_0t+\phi_0\right) + \frac{a_0a_1+\omega_va_2}{a_0^2-\omega_v^2}\xi_\perp\cos\left(\omega_vt+\delta\right), \\ \xi_\phi = \xi_\|\sin\left(a_0t+\phi_0\right) + \frac{a_0a_2+\omega_va_1}{a_0^2-\omega_v^2}\xi_\perp\sin\left(\omega_vt+\delta\right), \\ \xi_z = \xi_\perp - \left[\frac{a_0a_2+\omega_va_1}{a_0^2-\omega_v^2}\sin\left(a_0t+\phi_0\right)\sin\left(\omega_vt+\delta\right)\right. \\ \left. +\frac{a_0a_1+\omega_va_2}{a_0^2-\omega_v^2}\cos\left(a_0t+\phi_0\right)\cos\left(\omega_vt+\delta\right)\right]\xi_\|,\end{gathered} \tag{21}$$

where $\xi_\perp =< \xi_z(t) >$ and $\xi_\| = \sqrt{1-\xi_\perp^2}$. Angular brackets mean the time average. If $|a_0|$ is close to ω_v, the corrections to the spin dynamics given by Eq. (21) can be large. Therefore, the condition $|a_0| = \omega_v$ defines a resonance. When the spin rotation frequency is far from the resonance, Eqs. (16),(17),(19),(21) lead to the following expression for the average angular velocity of spin rotation:

$$< \dot{\psi} >= \lambda\Omega_0(1 - C_F), \quad C_F = \frac{1}{4}\left[1 - \frac{\Omega_0^2}{\gamma^2(\Omega_0^2 - \omega_v^2)} - \frac{\omega_v^2(f-1)(f-1+2/\gamma)}{\Omega_0^2 - \omega_v^2}\right]\psi_0^2. \quad (22)$$

This formula coincides with the result obtained in Refs. [7, 8]. However, in the real g−2 experiment $\omega_v/\Omega_0 \sim 10$. Therefore, we may not average over the spin rotation which is rather slow. As a result of averaging only over the vertical betatron oscillation, Eq. (22) takes the form

$$< \dot{\psi} >= \lambda\Omega_0(1 - C), \quad C = C_F + \frac{(\gamma-1)^2\Omega_0^2 - f^2\gamma^2\omega_v^2}{4\gamma^2(\Omega_0^2 - \omega_v^2)}\cos\left[2(\Omega_0 t + \lambda\phi_0)\right]\psi_0^2 \cdot \frac{1+\xi_\perp^2}{1-\xi_\perp^2}. \quad (23)$$

Since the average value of the last term oscillating with the angular frequency $2\Omega_0$ is zero, formula (23) is in the best agreement with the previously obtained result. However, it is reasonable to include the oscillatory term in the fitting process instead of eliminating it. In the real g−2 experiment, $f \approx 1$, $\xi_\perp = 0$, $\gamma \gg 1$, ϕ_0 equals $\pm\frac{\pi}{2}$, and the coefficient C is given by

$$C = \frac{1}{4}\left[1 - \cos\left(2\Omega_0 t\right)\right]\psi_0^2.$$

The quantities C_F and C are rather small because they are proportional to ψ_0^2. In the real g−2 experiment, ψ_0 is less than $1 \cdot 10^{-3}$.

The vertical betatron oscillation violates the sinusoidality of spin motion.

Essentially, the result of averaging over the vertical betatron oscillation is independent of an initial phase of this oscillation. Therefore, coherent and incoherent oscillations give the same effect. The oscillation of the angular velocity $< \dot{\psi} >$ averaged in such a way is the result of the fixed initial spin direction.

The quantity O in Eqs. (16) and (17) is much smaller than the corresponding quantity o in Eqs. (4) and (5). A significant difference between these quantities is conditioned by a relative smallness of N_ρ and ω_ϕ in comparison with ξ_ρ and $(\omega_a)_\phi$. Such a smallness is due to the fact that the radial component of particle momentum is proportional to the small quantity ρ_0 because it is caused by the radial oscillation. The radial component of spin given by Eq. (20) is defined by the g−2 precession. The importance of O has been first found by Granger and Ford [6] and formula (22) currently used has been derived by Farley [7].

VI. DISCUSSION

The particle EDM causes the spin rotation about the radial axis. As a result, the spin precession axis becomes tilted. If the spin rotation in the horizontal plane is cancelled or strongly restricted, the spin acquires a vertical component and the angle between the spin and the horizontal plane linearly increases from zero with time [14]. Therefore, the vertical magnetic field, the radial electric one, and magnetic focusing are expected to govern the particle motion in the EDM experiment. The best restriction of spin motion in the horizontal plane is obtained on condition that the radial electric field is adjusted to

$$E = E_0 = \frac{a\beta\gamma^2}{1 - a\beta^2\gamma^2}B$$

(see Ref. [3]). Such a restriction permits the high-precision detection of the EDM.

Field misalignments and beam oscillations cause the spin motion imitating the presence of the EDM. In particular, the electric field may not be perfectly in the horizontal plane. In this case, the appearance of a vertical electric field leads to the vertical deflection of spin simulating the EDM effect [3, 4]. The use of Eq. (15) simplifies the quantitative analysis of such effects. In this equation, the last but one term describes the spin rotation about the radial axis. The preceding terms also affect this rotation. The second-order effects discussed in Ref. [3] can also be calculated by means of Eq. (15). This equation determines the general approximate relation between the effects of AMM and EDM:

$$\frac{|\mathbf{\Omega}_{EDM}|}{|\boldsymbol{\omega}_a|} \approx \frac{\beta|\eta|}{2|a|} = \frac{\beta|d|}{|\mu'|}.$$

The EDM might cause a small tilt of the spin precession axis of muons in the g−2 experiment [2, 18]. Measuring this tilt sets a limit of 2.8×10^{-19} e·cm on the muon EDM [18]. If the muon EDM were equal to this limit, it would stimulate the rotation about the radial axis with the angular frequency $|\mathbf{\Omega}_{EDM}| = 3 \times 10^{-3}|\boldsymbol{\omega}_a|$.

The first excellent example of analytical description of complicated spin dynamics in storage rings is given in Refs. [6, 7, 8]. In these works, the effects of the vertical and radial CBOs on the horizontal motion of spin in the g−2 experiment are described. For this purpose, the particle rest frame was used. However, this frame is inconvenient for taking into account the synchrotron motion and the particle acceleration caused by a locally nonzero longitudinal electric field. For example, such a particle motion is important in the EDM experiment. If a strong radial electric field is used [3, 4], its imperfection results in the appearance of a longitudinal component leading to a systematical error. In Refs. [7, 8] and other works, the effect described by the term $o\boldsymbol{e}_z$ in Eq. (14) was ignored. The description of spin motion given in these works is therefore approximate, while Eq. (14) is exact.

Obtained equations (10),(14), and (15) describing the spin dynamics in cylindrical coordinates have been derived in the general form. They can be helpful for high-precision experiments at storage rings when the spin motion needs to be analytically calculated. Nevertheless, these equations are convenient only when the ring is either circular or divided into circular sectors by empty spaces. Moreover, the analytical description given by these equations can be useless if the configuration of the main fields is not simple enough. In this case, it is necessary to use computer calculations. The algorithms of such calculations can handle all kinds of common magnet types and also the effects of misalignments. These algorithms are based on different coordinate systems (e.g., on Frenet-Serret coordinates [5]). The method for calculating the spin motion in storage rings was developed by Derbenev, Kondratenko, and Skrinsky [19] (see also Ref. [20] and references therein). Another method using the Frenet-Serret curvilinear coordinates was proposed by Courant and Ruth [5]. These methods have been successfully used in various experiments and are also applicable in any other case.

However, the analytical description of spin dynamics can be necessary for several high-precision experiments. Only the analytical description can guarantee the clear understanding needed for the g−2 and EDM experiments. The use of Eqs. (10),(14), and (15) for analytical calculations of spin dynamics in these experiments can be very successful. Other formalisms seem to be less convenient. It is natural to use the cylindrical coordinate system for analyzing the spin motion in circular storage rings. Therefore, the obtained exact equation of spin motion in cylindrical coordinates can also be utilized in other cases where the configuration of the main fields is simple enough.

VII. SUMMARY

The exact formula for the angular velocity of particle motion in the horizontal plane has been obtained. The exact equation defining the spin motion in the cylindrical coordinate system with allowance for the particle EDM has been derived. This equation is convenient for analytical calculations of spin dynamics when the configuration of the main fields is simple enough. Such calculations can be needed for several high-precision experiments. Cylindrical coordinates can be used if the ring is either circular or divided into circular sectors. Averaging of the angular velocity of spin rotation has been performed. The generalized formula for the influence of the vertical betatron oscillation on the angular velocity of spin rotation in the g−2 experiment (pitch correction) has been found. This formula agrees with the previous result [7, 8] and contains the additional oscillatory term which can be used for fitting. The relative importance of the terms in the equation of spin motion including the EDM-dependent ones is discussed.

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