Analysis of wave equations for spin-1 particles interacting with an electromagnetic field

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Abstract

The Foldy-Wouthuysen transformation for relativistic spin-1 particles interacting with nonuniform electric and uniform magnetic fields is performed. The Hamilton operator in the Foldy-Wouthuysen representation is determined. It agrees with the Lagrangian obtained by Pomeransky, Khriplovich, and Sen'kov. The classical and quantum formulae for the Hamiltonian agree. The validity of the Corben-Schwinger equations is confirmed. However, it is difficult to generalize these equations in order to take into account the quadrupole moment defined by a particle charge distribution. The known second-order wave equations are not quite satisfactory because they contain non-Hermitian terms. The Hermitian second-order wave equation is derived.

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I. INTRODUCTION

The investigation of interaction of spin-1 particles with an electromagnetic field is very important for the high energy physics. There exist some difficulties in the spin-1 particle physics. There are many works where a consistency of spin-1 particle theories has been considered (see Ref. [1] and references therein). However, these works have not given us final conclusions. The method elaborated by Pomeransky, Khriplovich, and Sen'kov has made it possible to describe spin effects in an interaction of particles of any spin with an electromagnetic field [2, 3]. In particular, the Lagrangian with an allowance for second-order terms in spin has been calculated [2, 3]. In the present work, we use these results to verify some wave equations for spin-1 particles. We transform the Hamilton operator to the block-diagonal form (diagonal in two spinors) which defines the Foldy-Wouthuysen (FW) representation [4]. This representation is very convenient in order to analyze spin effects and perform the semiclassical transition. The obtained result is compared with both classical [5] and Pomeransky-Khriplovich-Sen'kov (PKS) [2, 3] approaches.

II. EQUATIONS FOR SPIN-1 PARTICLES

The situation in the spin-1 particle theory differs essentially from the situation in the spin-0 and spin-1/2 particle theories. The important difference is a great number of equations describing spin-1 particles (vector mesons). For the first time, the equations for vector mesons have been found by Proca [6]. For particles in an electromagnetic field, they have the form:

$$U_{\mu\nu} = D_\mu U_\nu - D_\nu U_\mu, \quad D^\mu U_{\mu\nu} = m^2 U_\nu, \quad D_\mu = \partial_\mu + ieA_\mu \equiv \left(\frac{\partial}{\partial t} + ie\Phi, \nabla - ie\boldsymbol{A}\right), \quad (1)$$

where A_μ, Φ, and $\boldsymbol{A}$ are 4-potential, scalar potential, and vector potential of electromagnetic field, respectively.

Spin-1 particles can be also described by the Duffin-Kemmer-Petiaux (DKP) [7, 8, 9], Stuckelberg [10], multispinor Bargmann-Wigner [11], and other equations. The DKP equation has the form

$$(\beta_\mu D_\mu + m)\psi = 0.$$

In this equation, the wave function ψ has ten components, and β_μ are 10×10 matrices.

They satisfy the conditions:

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\lambda} + \beta_\lambda\delta_{\mu\nu}.$$

Corben and Schwinger [12] showed how to include an anomalous magnetic dipole term in the Proca equations. Young and Bludman [13] took into account the additional electric quadrupole moment defined by a charge distribution in particles (charge quadrupole moment).

Many first-order equations are equivalent. They can be transformed one into another. This refers to the Proca, DKP, and Bargmann-Wigner equations (see Refs. [13, 14]). The Stuckelberg equations [10] differ essentially from other equations by the inclusion of an additional scalar field. As a result, the corresponding wave function has eleven components. The wave functions of the Proca and DKP equations have ten components.

Kahler [15] has proposed the equation for inhomogeneous differential forms which is equivalent to a system of scalar, vector, antisymmetric tensor, pseudovector and pseudoscalar fields [16, 17]. Kruglov [18] has generalized this equation on the case when the mass of scalar and pseudoscalar fields, m_0, differs from the mass of other fields, m. The corresponding wave function has sixteen components.

Several components of the Proca equations can be expressed in terms of the others. As a result, the equations for the ten-component wave function can be reduced to the equation for the six-component one (the generalized Sakata-Taketani equation [13, 19]). Since the components of the reduced wave function are two spinors, the wave function of the generalized Sakata-Taketani equation is a bispinor. This equation is very convenient for the semiclassical transition simplifying an investigation of spin dynamics.

Besides the first-order wave equations, there exist also second-order ones. These are the second-order forms of the above mentioned equations and some other equations (e.g., the Shay-Good equation [20]). The Shay-Good equation is not equivalent to the Proca theory.

III. CONSISTENCY OF SPIN-1 PARTICLE THEORIES

Soon after the appearance of the Proca theory, the problem of its consistency was stated [21]. There are many works where several difficulties of spin-1 particle theories have been investigated (e.g., complex energy modes for particles in a uniform magnetic field, see Refs.

[1, 22, 23, 24, 25] and references therein). In these works the problem of consistency of spin-1 particle theories was solved qualitatively. However, there exists a more exact criterium of validity of any particle theory. As is shown in Refs. [2, 3, 26], the spin motion of particles with arbitrary spin is described by the Bargmann-Michel-Telegdi (BMT) equation [27]:

$$\left(\frac{d\boldsymbol{S}}{dt}\right)_{BMT} = \frac{e}{2m}\left\{\left(g-2+\frac{2}{\gamma}\right)[\boldsymbol{S}\times\boldsymbol{B}] - (g-2)\frac{\gamma}{\gamma+1}[\boldsymbol{S}\times\boldsymbol{v}](\boldsymbol{v}\cdot\boldsymbol{B}) + \left(g-2+\frac{2}{\gamma+1}\right)[\boldsymbol{S}\times[\boldsymbol{E}\times\boldsymbol{v}]]\right\}, \tag{2}$$

where $\boldsymbol{S}$ is the spin operator, $\boldsymbol{E}$ is the electric field strength, and $\boldsymbol{B}$ is the magnetic field induction. The equation for the unit polarization vector, $\boldsymbol{O} =< \boldsymbol{S} > /S$, has the same form.

Any wave equation should be in congruence with the BMT equation.

To verify wave equations for spin-1 particles, the Lagrangian obtained in Refs. [2, 3] will also be used. This Lagrangian describes spin effects for particles of an arbitrary spin interacting with an electromagnetic field. It is given by

$$\begin{aligned}
&\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2,\\
&\mathcal{L}_1 = \frac{e}{2m}\left\{\left(g-2+\frac{2}{\gamma}\right)(\boldsymbol{S}\cdot\boldsymbol{B}) - (g-2)\frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{B}) + \left(g-2+\frac{2}{\gamma+1}\right)(\boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{v}])\right\},\\
&\mathcal{L}_2 = \frac{Q}{2S(2S-1)}\left[(\boldsymbol{S}\cdot\nabla) - \frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\nabla)\right]\left[(\boldsymbol{S}\cdot\boldsymbol{E}) - \frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{E}) + (\boldsymbol{S}\cdot[\boldsymbol{v}\times\boldsymbol{B}])\right] + \frac{e}{2m^2}\frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot[\boldsymbol{v}\times\nabla])\left[\left(g-1+\frac{1}{\gamma}\right)(\boldsymbol{S}\cdot\boldsymbol{B}) - (g-1)\frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{B}) + \left(g-\frac{\gamma}{\gamma+1}\right)(\boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{v}])\right],\\
&\gamma = \frac{1}{\sqrt{1-\boldsymbol{v}^2}} = \frac{\sqrt{m^2+\boldsymbol{\pi}^2}}{m},
\end{aligned} \tag{3}$$

where $g = 2\mu m/(eS)$, Q is the quadrupole moment and γ is the Lorentz factor. In Lagrangian (3), $\mathcal{L}_1$ contains terms that are linear in spin, while $\mathcal{L}_2$ contains quadratic terms. The Hermitian form of relation (3) is obtained by the substitution

$$\mathcal{L} \rightarrow (\mathcal{L} + \mathcal{L}^\dagger)/2.$$

The corresponding Hamiltonian equals this Lagrangian with the opposite sign: $\mathcal{H} = -\mathcal{L}$. In Eq. (3), the velocity operator is replaced by the corresponding classical quantity, and the spin is described by appropriate spin matrices. Therefore, Lagrangian (3) has been obtained in the semiclassical approximation. This Lagrangian has been used for finding the general equation of spin motion in nonuniform fields [28]:

$$\frac{d\boldsymbol{S}}{dt}=\left(\frac{d\boldsymbol{S}}{dt}\right)_{BMT}+\left(\frac{d\boldsymbol{S}}{dt}\right)_{q}, \tag{4}$$

$$\begin{aligned}\left(\frac{d\boldsymbol{S}}{dt}\right)_{q}=\frac{Q}{4S(2S-1)}\Biggl(\Biggl\{\left([\boldsymbol{S}\times\nabla]-\frac{\gamma}{\gamma+1}[\boldsymbol{S}\times\boldsymbol{v}](\boldsymbol{v}\cdot\nabla)\right),\Biggl((\boldsymbol{S}\cdot\boldsymbol{E})-\\
\frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{E})+(\boldsymbol{S}\cdot[\boldsymbol{v}\times\boldsymbol{B}])\Biggr)\Biggr\}_{+}+\Biggl\{\left((\boldsymbol{S}\cdot\nabla)-\frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\nabla)\right),\Biggl([\boldsymbol{S}\times\boldsymbol{E}]\\
-\frac{\gamma}{\gamma+1}[\boldsymbol{S}\times\boldsymbol{v}](\boldsymbol{v}\cdot\boldsymbol{E})+[\boldsymbol{S}\times[\boldsymbol{v}\times\boldsymbol{B}]]\Biggr)\Biggr\}_{+}\Biggr)\\
+\frac{e}{4m^2}\frac{\gamma}{\gamma+1}\Biggl(\Biggl\{[\boldsymbol{S}\times[\boldsymbol{v}\times\nabla]],\Biggl[\left(g-1+\frac{1}{\gamma}\right)(\boldsymbol{S}\cdot\boldsymbol{B})-(g-1)\frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{B})\\
+\left(g-\frac{\gamma}{\gamma+1}\right)(\boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{v}])\Biggr]\Biggr\}_{+}+\Biggl\{\left(\boldsymbol{S}\cdot[\boldsymbol{v}\times\nabla]\right),\Biggl[\left(g-1+\frac{1}{\gamma}\right)[\boldsymbol{S}\times\boldsymbol{B}]\\
-(g-1)\frac{\gamma}{\gamma+1}[\boldsymbol{S}\times\boldsymbol{v}](\boldsymbol{v}\cdot\boldsymbol{B})+\left(g-\frac{\gamma}{\gamma+1}\right)[\boldsymbol{S}\times[\boldsymbol{E}\times\boldsymbol{v}]]\Biggr]\Biggr\}_{+}\Biggr),\end{aligned} \tag{5}$$

where $\{\dots,\dots\}_+$ means an anticommutator.

The average product $< S_iS_j >$ is not equal to $< S_i >< S_j >$. The quantities $\left(\frac{d\boldsymbol{S}}{dt}\right)_{BMT}$ and $\left(\frac{d\boldsymbol{S}}{dt}\right)_{q}$ characterize the spin motion determined by the terms linear [BMT equation (2)] and quadratic [Eq. (5)] in spin, respectively. Lagrangian (3) and Eqs. (2),(4),(5) characterize the semiclassical approximation for the multispinor theory.

To verify any wave equation, it is helpful to transform it to the Hamilton form and then fulfil the semiclassical transition. The usual method of performing such a transition is the FW transformation [4]. The comparison of obtained semiclassical expressions with Eqs. (2)–(5) ensures a good possibility of the verification.

In the FW representation, the Hamiltonian and all the operators are block-diagonal (diagonal in two spinors). The relations between the operators are similar to those between the respective classical quantities. In this representation, the operators have the same form as in the nonrelativistic quantum theory. Only the FW representation possesses these properties considerably simplifying the transition to the semiclassical description. The FW representation provides the best possibility of obtaining a meaningful classical limit of the relativistic quantum theory [29].

In the FW representation, the polarization operator has the simplest form

$$\boldsymbol{\Pi} = \begin{pmatrix} \boldsymbol{S} & 0 \\ 0 & -\boldsymbol{S} \end{pmatrix},$$

where $\boldsymbol{S}$ is the 3×3 spin matrix for spin-1 particles. In other representations this operator is expressed by much more cumbersome formulae. Therefore, other representations are much less convenient in order to find spin motion equations. This conclusion is valid for particles of any spin.

For spin-1/2 particles, the polarization operator also takes the simplest form in the FW representation and cumbersome forms in other representations. The explicit expressions for this operator in the Dirac and FW representations are given in Refs. [30, 31].

The operator equation of spin motion is determined by the commutator

$$\frac{d\boldsymbol{\Pi}}{dt} = i[\mathcal{H}, \boldsymbol{\Pi}]. \tag{6}$$

To find the equation for the average spin, Eq. (6) should be averaged.

IV. FOLDY-WOUTHUYSEN TRANSFORMATION FOR SPIN-1 PARTICLES

The FW transformation for spin-1 particles has some peculiarities. The wave functions are pseudo-orthogonal, e.g., their normalization is defined by the relation

$$\int \Psi^\dagger \rho_3 \Psi dV = \int (\phi^\dagger \phi - \chi^\dagger \chi) dV = 1,$$

where $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ is the six-component wave function (bispinor). Here and below ρ_i ($i = 1, 2, 3$) are the Pauli matrices:

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Components of these matrices act on the upper and lower spinors. The Hamiltonian for spin-1 particles is pseudo-Hermitian, that is, it satisfies the conditions

$$\mathcal{H} = \rho_3 \mathcal{H}^\dagger \rho_3, \quad \mathcal{H}^\dagger = \rho_3 \mathcal{H} \rho_3.$$

Even (diagonal) terms of the Hamiltonian are Hermitian and odd (off-diagonal) terms are anti-Hermitian.

The operator U transforming the wave function to a different representation should be pseudo-unitary:

$$U^{-1} = \rho_3 U^\dagger \rho_3, \quad U^\dagger = \rho_3 U^{-1} \rho_3.$$

The transformed Hamiltonian equals [4]:

$$\mathcal{H}' = U\left(\mathcal{H} - i\frac{\partial}{\partial t}\right)U^{-1} + i\frac{\partial}{\partial t}.$$

The initial Hamiltonian is determined by the generalized Sakata-Taketani equation which can be written in the form

$$\mathcal{H} = \rho_3\mathcal{M} + \mathcal{E} + \mathcal{O}, \quad \rho_3\mathcal{E} = \mathcal{E}\rho_3, \quad \rho_3\mathcal{O} = -\mathcal{O}\rho_3, \tag{7}$$

where $\mathcal{E}$ and $\mathcal{O}$ are the even and odd operators, commuting and anticommuting with ρ_3, respectively.

When

$$[\mathcal{M}, \mathcal{O}] = 0, \quad [\mathcal{E}, \mathcal{O}] = 0, \tag{8}$$

and the external field is stationary, the exact transformation of Hamiltonian $\mathcal{H}$ to the block-diagonal form can be fulfilled with the operator

$$U = \frac{\epsilon + \mathcal{M} + \rho_3\mathcal{O}}{\sqrt{2\epsilon(\epsilon + \mathcal{M})}}, \quad U^{-1} = \frac{\epsilon + \mathcal{M} - \rho_3\mathcal{O}}{\sqrt{2\epsilon(\epsilon + \mathcal{M})}}, \quad \epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2}. \tag{9}$$

For spin-1/2 particles, the similar property has been proved in Ref. [31].

The transformed Hamiltonian is equal to

$$\mathcal{H}' = \rho_3\epsilon + \mathcal{E}.$$

In the general case, the external field is not stationary and the operator $\mathcal{O}$ commutes neither with $\mathcal{M}$ nor with $\mathcal{E}$. In this case the following transformation method can be used. The operator $\mathcal{O}$ can be divided into two operators:

$$\mathcal{O} = \mathcal{O}_1 + \mathcal{O}_2. \tag{10}$$

The operator $\mathcal{O}_1$ should commute with $\mathcal{M}$ and the operator $\mathcal{O}_2$ should be equal to zero for the free particle. Therefore, the operator $\mathcal{O}_2$ should be relatively small.

First, it is necessary to perform the unitary transformation with the operator

$$U=\frac{\epsilon+\mathcal{M}+\rho_3\mathcal{O}_1}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}},\quad U^{-1}=\frac{\epsilon+\mathcal{M}-\rho_3\mathcal{O}_1}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}},\quad \epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}_1^2}. \tag{11}$$

After this transformation, the Hamiltonian $\mathcal{H}'$ still contains odd terms proportional to the derivatives of the potentials. The operator $\mathcal{H}'$ can be written in the form

$$\mathcal{H}'=\rho_3\epsilon+\mathcal{E}'+\mathcal{O}',\quad \rho_3\mathcal{E}'=\mathcal{E}'\rho_3,\quad \rho_3\mathcal{O}'=-\mathcal{O}'\rho_3, \tag{12}$$

where ϵ is defined by Eq. (11). The odd operator $\mathcal{O}'$ is small compared to both ϵ and the initial Hamiltonian $\mathcal{H}$. This circumstance allows us to apply the usual scheme of the nonrelativistic FW transformation [4, 31, 32].

Second, the transformation should be performed with the following operator:

$$U'=\exp{(iS')},\quad S'=-\frac{i}{4}\rho_3\left\{\mathcal{O}',\frac{1}{\epsilon}\right\}_+=-\frac{i}{4}\left[\frac{\rho_3}{\epsilon},\mathcal{O}'\right]. \tag{13}$$

The further calculations are similar to those performed for spin-1/2 particles [31, 32]. As compared with Ref. [32], the particle mass should be replaced by the operator ϵ noncommuting with the operators $\mathcal{E}'$ and $\mathcal{O}'$. If only major corrections are taken into account, then the transformed Hamiltonian equals

$$\mathcal{H}''=\rho_3\epsilon+\mathcal{E}'+\frac{1}{4}\rho_3\left\{\frac{1}{\epsilon},\mathcal{O}'^2\right\}_+. \tag{14}$$

This is the Hamiltonian in the FW representation.

To obtain the desired accuracy, the calculation procedure with transformation operator (13) (S' is replaced by S'', S''', etc.) should be repeated multiply.

It is important that the diagonalization of Hamiltonian is not equivalent to the FW transformation. There exists an infinite set of transformations resulting in block-diagonal forms of all the operators. Therefore, the equivalence of any representation to the FW one should be verified. For spin-1/2 particles, the example of the diagonalization which does not lead to the FW representation has been shown in Ref. [33]. The similar situation takes place for spin-1 particles. In particular, the transformation performed by Roux [34] does not lead to the FW representation either.

V. HAMILTONIAN FOR SPIN-1 PARTICLES IN A NONUNIFORM ELECTROMAGNETIC FIELD

Young and Bludman [13] have included terms describing the charge quadrupole moment of particles in the Corben-Schwinger (CS) equations [12] and have made the Sakata-Taketani transformation [19]. The generalized Sakata-Taketani equation obtained in Ref. [13] defines the Hamiltonian acting on the six-component bispinor. This equation is similar to the Dirac equation for spin-1/2 particles. Therefore, it is useful to perform the FW transformation. In this section, we make such a transformation without an allowance for the charge quadrupole moment of particles.

The above described method is used for finding the transformed Hamiltonian to within second-order terms in the field potentials and first-order terms in the field strengths and first-order derivatives of the electric field strength. The terms of the second order and higher in the field strengths and their derivatives and the first-order terms containing derivatives of all the orders of the magnetic field strength and derivatives of the second order and higher of the electric field strength will be omitted. The external fields are considered to be stationary.

In this approximation, the basic generalized Sakata-Taketani equation for the Hamiltonian takes the form [13]

$$\begin{gathered}\mathcal{H}=e\Phi+\rho_3 m+i\rho_2\frac{1}{m}(\boldsymbol{S}\cdot\boldsymbol{D})^2-(\rho_3+i\rho_2)\frac{1}{2m}(\boldsymbol{D}^2+e\boldsymbol{S}\cdot\boldsymbol{H})-\\ (\rho_3-i\rho_2)\frac{e\kappa}{2m}(\boldsymbol{S}\cdot\boldsymbol{H})-\frac{e\kappa}{2m^2}(1+\rho_1)\Bigl[(\boldsymbol{S}\cdot\boldsymbol{E})(\boldsymbol{S}\cdot\boldsymbol{D})-i\boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{D}]-\boldsymbol{E}\cdot\boldsymbol{D}\Bigr]+\\ \frac{e\kappa}{2m^2}(1-\rho_1)\Bigl[(\boldsymbol{S}\cdot\boldsymbol{D})(\boldsymbol{S}\cdot\boldsymbol{E})-i\boldsymbol{S}\cdot[\boldsymbol{D}\times\boldsymbol{E}]-\boldsymbol{D}\cdot\boldsymbol{E}\Bigr],\end{gathered}\tag{15}$$

where $\boldsymbol{H}$ is the magnetic field strength, $\kappa=$const, and $\boldsymbol{D}=\nabla-ie\boldsymbol{A}$.

This equation satisfies Eqs. (7),(10), if

$$\begin{gathered}\mathcal{M}=m+\frac{\boldsymbol{\pi}^2}{2m}-\frac{e}{m}\boldsymbol{S}\cdot\boldsymbol{H},\\ \mathcal{E}=e\Phi-\rho_3\frac{e(\kappa-1)}{2m}\boldsymbol{S}\cdot\boldsymbol{H}+\\ \frac{e\kappa}{4m^2}\Bigl(\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]-\boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}]+\{\boldsymbol{S}\cdot\nabla,\boldsymbol{S}\cdot\boldsymbol{E}\}_+-2\nabla\cdot\boldsymbol{E}\Bigr),\\ \mathcal{O}_1=i\rho_2\left[\frac{\boldsymbol{\pi}^2}{2m}-\frac{(\boldsymbol{\pi}\cdot\boldsymbol{S})^2}{m}+\frac{e(\kappa-1)}{2m}\boldsymbol{S}\cdot\boldsymbol{H}\right],\\ \mathcal{O}_2=i\rho_1\frac{e\kappa}{2m^2}\Bigl(\boldsymbol{\pi}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot\boldsymbol{\pi}-\{\boldsymbol{S}\cdot\boldsymbol{\pi},\boldsymbol{S}\cdot\boldsymbol{E}\}_++\boldsymbol{S}\cdot[\nabla\times\boldsymbol{E}]\Bigr),\end{gathered}\tag{16}$$

where $\boldsymbol{\pi} = -i\boldsymbol{D} = -i\nabla - e\boldsymbol{A}$ is the kinetic momentum operator and $\{\boldsymbol{S}\cdot\nabla, \boldsymbol{S}\cdot\boldsymbol{E}\}_+ \equiv (S_iS_j + S_jS_i)(\partial E_i/\partial x_j)$.

The pseudo-unitary FW transformation leads to Eq. (12) where

$$\begin{gathered}
\epsilon = \epsilon' - \left\{\frac{e}{2\epsilon'}, \boldsymbol{S}\cdot\boldsymbol{H}\right\}_+ + \frac{e(\kappa-1)}{16m^2}\left\{\frac{1}{\epsilon'}, \{\boldsymbol{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\boldsymbol{H}\}_+\right\}_+, \\
\mathcal{E}' = e\Phi + \frac{e}{4m}\left[\left\{\left(\frac{\kappa-1}{2} + \frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'}, (\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}] - \boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])\right\}_+ - \right. \\
\left. 2\rho_3(\kappa-1)\boldsymbol{S}\cdot\boldsymbol{H} - \rho_3\left\{\frac{(\kappa-1)(\epsilon'-m)}{4m\epsilon'(\epsilon'+m)}, \{\boldsymbol{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\boldsymbol{H}\}_+\right\}_+\right] + \\
\frac{e\kappa}{4m^2}\left\{\left(\boldsymbol{S}\cdot\nabla - \frac{1}{\epsilon'(\epsilon'+m)}(\boldsymbol{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\nabla)\right), \left(\boldsymbol{S}\cdot\boldsymbol{E} - \frac{1}{\epsilon'(\epsilon'+m)}(\boldsymbol{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{E})\right)\right\}_+ + \\
\frac{e}{8m^2}\left\{\frac{1}{\epsilon'(\epsilon'+m)}\left(\kappa + \frac{m}{\epsilon'+m}\right), \left\{\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\nabla], \boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]\right\}_+\right\}_+ - \\
\frac{e\kappa}{2m^2}\nabla\cdot\boldsymbol{E} + \frac{e}{4m^2}\left\{\frac{1}{\epsilon'^2}\left(\kappa + \frac{m^2}{4\epsilon'^2}\right), (\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})\right\}_+, \quad \epsilon' = \sqrt{m^2+\boldsymbol{\pi}^2}.
\end{gathered} \tag{17}$$

In this equation the operator $\mathcal{O}'$ is proportional to the field strengths. After the second transformation, the contribution of this operator to the Hamiltonian $\mathcal{H}''$ is proportional to $\mathcal{O}'^2$. Such a contribution is negligible and the Hamiltonian in the FW representation equals

$$\mathcal{H}'' = \rho_3\epsilon + \mathcal{E}' \tag{18}$$

or

$$\begin{gathered}
\mathcal{H}'' = \rho_3\epsilon' + e\Phi + \frac{e}{4m}\left[\left\{\left(\frac{\kappa-1}{2} + \frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'}, (\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}] - \boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])\right\}_+ - \right. \\
\left. \rho_3\left\{\left(\kappa - 1 + \frac{2m}{\epsilon'}\right), \boldsymbol{S}\cdot\boldsymbol{H}\right\}_+ + \rho_3\left\{\frac{\kappa-1}{2\epsilon'(\epsilon'+m)}, \{\boldsymbol{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\boldsymbol{H}\}_+\right\}_+\right] + \\
\frac{e\kappa}{4m^2}\left\{\left(\boldsymbol{S}\cdot\nabla - \frac{1}{\epsilon'(\epsilon'+m)}(\boldsymbol{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\nabla)\right), \left(\boldsymbol{S}\cdot\boldsymbol{E} - \frac{1}{\epsilon'(\epsilon'+m)}(\boldsymbol{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{E})\right)\right\}_+ + \\
\frac{e}{8m^2}\left\{\frac{1}{\epsilon'(\epsilon'+m)}\left(\kappa + \frac{m}{\epsilon'+m}\right), \left\{\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\nabla], \boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]\right\}_+\right\}_+ - \\
\frac{e\kappa}{2m^2}\nabla\cdot\boldsymbol{E} + \frac{e}{4m^2}\left\{\frac{1}{\epsilon'^2}\left(\kappa + \frac{m^2}{4\epsilon'^2}\right), (\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})\right\}_+.
\end{gathered} \tag{19}$$

We can introduce the g factor to describe the anomalous magnetic moment (AMM). In this case, $g=\kappa+1$ and the Hamiltonian takes the form

$$\mathcal{H}''=\rho_3\epsilon'+e\Phi+\frac{e}{4m}\Biggl[\left\{\left(\frac{g-2}{2}+\frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'},(\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]-\boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])\right\}_+ -$$
$$\rho_3\left\{\left(g-2+\frac{2m}{\epsilon'}\right),\boldsymbol{S}\cdot\boldsymbol{H}\right\}_+ +\rho_3\left\{\frac{g-2}{2\epsilon'(\epsilon'+m)},\{\boldsymbol{S}\cdot\boldsymbol{\pi},\boldsymbol{\pi}\cdot\boldsymbol{H}\}_+\right\}_+\Biggr]+$$
$$\frac{e(g-1)}{4m^2}\left\{\left(\boldsymbol{S}\cdot\nabla-\frac{1}{\epsilon'(\epsilon'+m)}(\boldsymbol{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\nabla)\right),\left(\boldsymbol{S}\cdot\boldsymbol{E}-\frac{1}{\epsilon'(\epsilon'+m)}(\boldsymbol{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{E})\right)\right\}_+ +$$
$$\frac{e}{8m^2}\left\{\frac{1}{\epsilon'(\epsilon'+m)}\left(g-1+\frac{m}{\epsilon'+m}\right),\left\{\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\nabla],\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]\right\}_+\right\}_+ -$$
$$\frac{e(g-1)}{2m^2}\nabla\cdot\boldsymbol{E}+\frac{e}{4m^2}\left\{\frac{1}{\epsilon'^2}\left(g-1+\frac{m^2}{4\epsilon'^2}\right),(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})\right\}_+ . \tag{20}$$

The g factor of $g=g_{Pr}=1$ corresponds to the Proca particle. Nevertheless, the preferred g factor is equal to 2 [2, 3, 35].

The more particular case of the uniform electric and magnetic fields has been considered in Ref. [28]. Within the nonrelativistic limit, formula (20) agrees with the result obtained in Ref. [13]. For the case of nonrelativistic particles in the magnetic field, this formula complies with the Hamiltonian derived by Case [36], who also used the FW transformation.

Since the electric field is considered to be stationary, $\boldsymbol{E}=-\nabla\Phi$. Therefore, the operators $\boldsymbol{S}\cdot\nabla$ and $\boldsymbol{S}\cdot\boldsymbol{E}$ commute. In any case, ∇ operates on $\boldsymbol{E}$.

The transition to the semiclassical approximation consists in averaging the Hamilton operator over the wave functions of stationary states. For free particles, the lower spinor is equal to zero in the FW representation. For particles in external fields, the maximum ratio of the lower and upper spinors is of the first order in W_{int}/E, where W_{int} is the energy of the particle interaction with external fields and E is the total energy of a particle. Thus, we obtain $(\chi^\dagger\chi)/(\phi^\dagger\phi)\sim(W_{int}/E)^2$ [31]. Therefore, the contribution of the lower spinor is negligible and the transition to the semiclassical equation is performed by averaging the operators in the equation for the upper spinor. It is usually possible to neglect the commutators between the coordinate and kinetic momentum operators and between different components of the kinetic momentum operator (see Ref. [37]). As a result, the operator $\boldsymbol{\pi}$ should be substituted by the classical kinetic momentum. For this classical quantity we retain the same designation. The semiclassical Hamiltonian is expressed by the relation

$$\begin{gathered}
\mathcal{H}'' = \epsilon' + e\Phi + \frac{e}{2m}\left[\left(g-2+\frac{2}{\gamma+1}\right)\Bigl(\boldsymbol{S}\cdot[\boldsymbol{v}\times\boldsymbol{E}]\Bigr) - \right.\\
\left.\left(g-2+\frac{2}{\gamma}\right)\boldsymbol{S}\cdot\boldsymbol{H} + \frac{(g-2)\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{H}\}\right] + \\
\frac{e(g-1)}{2m^2}\left[\boldsymbol{S}\cdot\nabla - \frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\nabla)\right]\left[\boldsymbol{S}\cdot\boldsymbol{E} - \frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{E})\right] + \\
\frac{e\gamma}{2m^2(\gamma+1)}\left(g-1+\frac{1}{\gamma+1}\right)\Bigl(\boldsymbol{S}\cdot[\boldsymbol{v}\times\nabla]\Bigr)\Bigl(\boldsymbol{S}\cdot[\boldsymbol{v}\times\boldsymbol{E}]\Bigr) - \\
\frac{e(g-1)}{2m^2}\nabla\cdot\boldsymbol{E} + \frac{e}{2m^2}\left(g-1+\frac{1}{4\gamma^2}\right)(\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}\cdot\boldsymbol{E}),
\end{gathered} \tag{21}$$

where $\gamma = \epsilon'/m$ is the Lorentz factor and $\boldsymbol{v} = \boldsymbol{\pi}/\epsilon'$ is the velocity. This relation is in the best compliance with formula (3) defining the Lagrangian for particles of an arbitrary spin.

Formulae (19)–(21) contain spin-independent terms proportional to the derivatives of $\boldsymbol{E}$. These terms have not been calculated in [2, 3].

The perfect agreement between Hamiltonian (21) and the Lagrangian derived in Refs. [2, 3] causes such an agreement between the corresponding equations of spin motion. As a result of the semiclassical transition, the particle polarization operator reduces to the matrix $\boldsymbol{S}$. The spin motion equation has the form

$$\begin{gathered}
\frac{d\boldsymbol{S}}{dt} = \left(\frac{d\boldsymbol{S}}{dt}\right)_{BMT} + \left(\frac{d\boldsymbol{S}}{dt}\right)_q, \\
\left(\frac{d\boldsymbol{S}}{dt}\right)_q = \frac{Q}{2}\left[\left([\boldsymbol{S}\times\nabla] - \frac{\gamma}{\gamma+1}[\boldsymbol{S}\times\boldsymbol{v}](\boldsymbol{v}\cdot\nabla)\right)\left((\boldsymbol{S}\cdot\boldsymbol{E}) - \frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{E})\right) + \right.\\
\left.\left((\boldsymbol{S}\cdot\nabla) - \frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\nabla)\right)\left([\boldsymbol{S}\times\boldsymbol{E}] - \frac{\gamma}{\gamma+1}[\boldsymbol{S}\times\boldsymbol{v}](\boldsymbol{v}\cdot\boldsymbol{E})\right)\right] + \\
\frac{e}{2m^2}\frac{\gamma}{\gamma+1}\left(g-\frac{\gamma}{\gamma+1}\right)\Bigl([\boldsymbol{S}\times[\boldsymbol{v}\times\nabla]]\,(\boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{v}]) + (\boldsymbol{S}\cdot[\boldsymbol{v}\times\nabla])\,[\boldsymbol{S}\times[\boldsymbol{E}\times\boldsymbol{v}]]\Bigr), \\
Q = -\frac{e(g-1)}{m^2},
\end{gathered} \tag{22}$$

where $\left(\frac{d\boldsymbol{S}}{dt}\right)_{BMT}$ is given by Eq. (2) and Q is the quadrupole moment.

Thus, a rigorous calculation shows that, upon the FW transformation, the Hamiltonian determined on the basis of the Proca theory (with an allowance for AMM [12]) is fully consistent with the PKS theory [2, 3]. The spin motion equation agrees with the corresponding equation obtained in Ref. [28]. Therefore, the Proca and CS equations correctly describe, at least, weak-field effects.

VI. CHARGE QUADRUPOLE MOMENT OF PARTICLES

Spin-1 particles can possess the charge quadrupole moment. Terms describing such a moment can be added to the Lagrangian [13]. They should be bilinear in the meson field variables U_μ and $U_{\mu\nu}$ and linear in the derivatives of the electromagnetic field $\partial_\lambda F_{\mu\nu}$. The choice of these terms is strongly restricted by the Maxwell equations. As a result, there exists the only form of extra terms describing the charge quadrupole moment of particles [13].

The terms that can be added to initial generalized Sakata-Taketani Hamiltonian (15) are given by

$$\Delta\mathcal{H} = \frac{eq}{4m^2}\left[(S_iS_j + S_jS_i)\frac{\partial E_i}{\partial x_j} - 2\frac{\partial E_i}{\partial x_i}\right] \equiv \frac{eq}{4m^2}\left[\{(\boldsymbol{S}\cdot\nabla),(\boldsymbol{S}\cdot\boldsymbol{E})\}_+ - 2\nabla\cdot\boldsymbol{E}\right], \qquad (23)$$

where q=const. These terms should be included into the operator $\mathcal{E}$. The operator of the unitary transformation defined by Eq. (11) remains unchanged. As a result of the FW transformation, Hamilton operator (20) should be added by the terms

$$\Delta\mathcal{H}'' = \frac{eq}{2m^2}\Biggl[(\boldsymbol{S}\cdot\nabla)(\boldsymbol{S}\cdot\boldsymbol{E}) - \frac{1}{\epsilon' m(\epsilon'+m)^2}(\boldsymbol{S}\cdot\boldsymbol{\pi})^2(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E}) + \frac{\epsilon'-m}{4\epsilon' m(\epsilon'+m)}\left(\Bigl\{\boldsymbol{S}\cdot\boldsymbol{\pi},(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{S}\cdot\boldsymbol{E})\Bigr\}_+ + \Bigl\{\boldsymbol{S}\cdot\boldsymbol{\pi},(\boldsymbol{S}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})\Bigr\}_+\right) - \nabla\cdot\boldsymbol{E}\Biggr]. \qquad (24)$$

In the nonrelativistic approximation,

$$\Delta\mathcal{H}'' = \frac{eq}{2m^2}\Bigl[(\boldsymbol{S}\cdot\nabla)(\boldsymbol{S}\cdot\boldsymbol{E}) - \nabla\cdot\boldsymbol{E}\Bigr]. \qquad (25)$$

This formula is in agreement with the result obtained in Ref. [13]. The Hamiltonian describing the quadrupole interaction of nonrelativistic spin-1 particles is given by

$$\mathcal{H}_q = -\frac{1}{6}Q_{ij}\frac{\partial E_i}{\partial x_j}, \quad Q_{ij} = \frac{3}{2}Q(S_iS_j + S_jS_i - \frac{4}{3}\delta_{ij}), \qquad (26)$$

where Q is the quadrupole moment. With an allowance for Eqs. (22),(25), it is equal to

$$Q = -\frac{e(g-1+q)}{m^2}.$$

The forms of Eqs. (20) and (24) are very different. In the semiclassical approximation, Hamiltonian (24) is expressed by the relation

$$\Delta\mathcal{H}'' = -\frac{Q}{2}\Biggl[(\boldsymbol{S}\cdot\nabla)(\boldsymbol{S}\cdot\boldsymbol{E}) - \frac{\gamma^3}{(\gamma+1)^2}(\boldsymbol{S}\cdot\boldsymbol{v})^2(\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}\cdot\boldsymbol{E}) + \frac{\gamma(\gamma-1)}{4(\gamma+1)}\left(\Bigl\{\boldsymbol{S}\cdot\boldsymbol{v},(\boldsymbol{v}\cdot\nabla)(\boldsymbol{S}\cdot\boldsymbol{E})\Bigr\}_+ + \Bigl\{\boldsymbol{S}\cdot\boldsymbol{v},(\boldsymbol{S}\cdot\nabla)(\boldsymbol{v}\cdot\boldsymbol{E})\Bigr\}_+\right) - \nabla\cdot\boldsymbol{E}\Biggr], \qquad (27)$$

where $Q = -eq/m^2$. Formula (27), unlike formula (21), disagrees with the relativistic expression for the Lagrangian obtained in Refs. [2, 3].

It is evident that formulae (24),(27) does not agree with formulae (20),(21). The classical description of the quadrupole interaction of relativistic particles has been given in Ref. [5]. The results obtained in this work are in agreement with formulae (20),(21) and contradict formulae (24),(27).

The disagreement between the formulae for the charge quadrupole moment obtained by different methods poses a difficult problem. Young and Bludman [13] used the approach based on an inclusion of appropriate terms into the first-order Proca Lagrangian. An addition of first-order terms to the Proca Lagrangian results in the CS equations which are correct. It is important that there exist the only form of second-order terms describing the charge quadrupole moment of particles. However, an inclusion of these terms in the Lagrangian does not results in a correct relativistic description of the charge quadrupole moment. As opposed to the Young-Bludman approach, the PKS one [2, 3] leads to the correct second-order Lagrangian. The comparison of this Lagrangian with the classical formulae obtained in Ref. [5] confirms its validity.

VII. COMPARISON OF CLASSICAL AND QUANTUM FORMULAE FOR THE HAMILTONIAN

In the classical theory, the spin motion is usually described by the well-known Good-Nyborg (GN) equation [38, 39]. For the above used designations, the three-dimensional form of this equation is given by [38]

$$\begin{gathered}\frac{d\boldsymbol{O}}{dt}=\left(\frac{d\boldsymbol{O}}{dt}\right)_{BMT}+\left(\frac{d\boldsymbol{O}}{dt}\right)_{G}, \qquad \boldsymbol{O}=\frac{<\boldsymbol{S}>}{S},\\ \left(\frac{d\boldsymbol{O}}{dt}\right)_{G}=\frac{Q}{2S-1}\left((\boldsymbol{O}\cdot\nabla)+\frac{\gamma^2}{\gamma+1}(\boldsymbol{O}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\nabla)\right)\Biggl([\boldsymbol{O}\times\boldsymbol{E}]-\\ \frac{\gamma}{\gamma+1}[\boldsymbol{O}\times\boldsymbol{v}](\boldsymbol{v}\cdot\boldsymbol{E})+[\boldsymbol{O}\times[\boldsymbol{v}\times\boldsymbol{B}]]\Biggr)+\\ \frac{egS}{2m^2}\frac{\gamma}{\gamma+1}\Bigl[\boldsymbol{O}\times[\boldsymbol{v}\times\nabla]\Bigr]\left[(\boldsymbol{O}\cdot\boldsymbol{B})-\frac{\gamma}{\gamma+1}(\boldsymbol{O}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{B})+\Bigl(\boldsymbol{O}\cdot[\boldsymbol{E}\times\boldsymbol{v}]\Bigr)\right],\end{gathered} \tag{28}$$

where $\left(\frac{d\boldsymbol{O}}{dt}\right)_{BMT}$ is expressed by the BMT equation.

It is obvious that Eq. (28) disagrees with Eqs. (5) and (22). A possible reason has been pointed out in Refs. [40, 41]. In Refs. [38, 39] and some other works the condition of orthogonality of four-vectors of velocity and polarization ($u^\nu a_\nu = 0$) has been used. However, this condition means the polarization vector is defined in the particle rest frame [40]. This frame is accelerated. Defining the polarization vector of particle in the rest frame instead of the instantly accompanying frame results in changing the spin motion equation [40]. In this case, further calculations are not well-grounded.

In any case, the use of the Lagrangian or Hamiltonian for defining the classical equation of spin motion is quite possible. Such an approach has been used in Refs. [5, 40, 41]. The Lorentz contraction of longitudinal sizes of moving bodies changes the Hamiltonian that takes the form

$$\mathcal{H} = \frac{1}{6}Q_{ij}\frac{\partial^2\phi}{\partial x_i \partial x_j} + \frac{1}{6}\tau\frac{\partial^2\phi}{\partial x_i^2}, \tag{29}$$

where Q_{ij} is the tensor of quadrupole moment, τ is the root-mean-square charge radius, and x_i $(i = 1, 2, 3)$ are the coordinates of a moving particle. For nonrelativistic particles $Q_{ij} = Q_{ij}^{(0)}$, $\tau = \tau^{(0)}$, where

$$Q_{ij}^{(0)} = \frac{3Q}{2S(2S-1)}\left[S_iS_j + S_jS_i - \frac{2}{3}S(S+1)\delta_{ij}\right] \tag{30}$$

and S_i $(i = 1, 2, 3)$ are the spin components. For relativistic particles [5]

$$\begin{gathered}
Q_{ij} = Q_{ij}^{(0)} - \frac{\gamma}{\gamma+1}\left(v_iv_kQ_{kj}^{(0)} + v_jv_kQ_{ki}^{(0)}\right) + \frac{\gamma^2}{(\gamma+1)^2}v_iv_jv_kv_lQ_{lk}^{(0)} + \\
+\frac{1}{3}\delta_{ij}v_kv_lQ_{lk}^{(0)} - \left(v_iv_j - \frac{1}{3}\delta_{ij}v^2\right)\tau^{(0)}, \quad \tau = \left(1 - \frac{1}{3}v^2\right)\tau^{(0)} - \frac{1}{3}v_kv_lQ_{lk}^{(0)}.
\end{gathered} \tag{31}$$

The corresponding Hamiltonian takes the form

$$\begin{gathered}
\mathcal{H} = -\frac{Q}{2S(2S-1)}\left[\boldsymbol{S}\cdot\nabla - \frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\nabla)\right]\left[\boldsymbol{S}\cdot\boldsymbol{E} - \frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{E})\right] - \\
\frac{1}{6}\left[\tau_0 - Q\frac{S+1}{2S-1}\right]\left[\nabla\cdot\boldsymbol{E} - (\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}\cdot\boldsymbol{E})\right].
\end{gathered} \tag{32}$$

Classical formula (32) agrees with quantum formulae (3) and (21). Formulae (3),(21), and (32) give the same relativistic dependence of terms proportional to the first-order derivatives of the field strengths. This refers to the terms with and without the spin. The only difference is an absence of the term proportional to $\Bigl(\boldsymbol{S}\cdot[\boldsymbol{v}\times\nabla]\Bigr)\Bigl(\boldsymbol{S}\cdot[\boldsymbol{v}\times\boldsymbol{E}]\Bigr)$ in classical Hamiltonian (32). However, this term describes neither quadrupole interaction nor contact one. Therefore, it is of a purely quantum origin. Nevertheless, including the considered term in the classical equation of spin motion is not impossible.

The classical equation of spin motion differing from GN equation (28) has been derived in Ref. [5]. Use of Hamiltonian (32) makes it possible to rewrite this equation in the more compact form:

$$\begin{gathered}\frac{d\boldsymbol{S}}{dt}=\left(\frac{d\boldsymbol{S}}{dt}\right)_{BMT}+\left(\frac{d\boldsymbol{S}}{dt}\right)_{q},\\ \left(\frac{d\boldsymbol{S}}{dt}\right)_{q}=\frac{Q}{2}\left[\boldsymbol{S}\times\nabla-\frac{\gamma}{\gamma+1}[\boldsymbol{S}\times\boldsymbol{v}](\boldsymbol{v}\cdot\nabla)\right]\left[\boldsymbol{S}\cdot\boldsymbol{E}-\frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{E})\right]+\\ \frac{Q}{2}\left[\boldsymbol{S}\cdot\nabla-\frac{\gamma}{\gamma+1}(\boldsymbol{S}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\nabla)\right]\left[\boldsymbol{S}\times\boldsymbol{E}-\frac{\gamma}{\gamma+1}[\boldsymbol{S}\times\boldsymbol{v}](\boldsymbol{v}\cdot\boldsymbol{E})\right].\end{gathered}\tag{33}$$

Eq. (33) agrees with quantum equations (5),(22) and disagrees with GN equation (28).

Thus, the comparison of the quantum Lagrangian and Hamiltonian with the classical Hamiltonian derived in Ref. [5] confirms the validity of both the PKS approach and the CS equations. The GN equation is not satisfactory.

VIII. WAVE EQUATIONS OF THE SECOND ORDER

Second-order wave equations can be obtained with an elimination of several components of wave functions. For example, the second-order form of Proca equations (1) is the result of substitution of $U_{\mu\nu}$ into the second equation. After the substitution, the second-order Proca equation takes the form [6]

$$D^\mu D_\mu U_\nu - D^\mu D_\nu U_\mu = m^2 U_\nu. \tag{34}$$

As a rule, the elimination of several components changes properties of wave functions and wave equations. Wave equations become non-Hermitian. For example, Eq. (34) is non-Hermitian because

$$(D^\mu D_\nu)^\dagger = D_\nu D^\mu = D^\mu D_\nu + g^{\mu\rho}[D_\nu, D_\rho] = D^\mu D_\nu - ieg^{\mu\rho}F_{\rho\nu} \neq D^\mu D_\nu,$$

where $g^{\mu\rho}=\mathrm{diag}\{1,-1,-1,-1\}$ is the metric tensor and $F_{\rho\nu}$ is the tensor of electromagnetic field.

A majority of second-order wave equations for spin-1 particles has been obtained in this way. These equations are non-Hermitian. In the general case, non-Hermitian equations have complex eigenvalues and nonorthogonal eigenfunctions. Of course, the elimination of several components of wave functions changes neither residuary components nor energy modes.

However, only eigenfunctions of initial Hermitian wave equations are (pseudo)orthogonal. After the elimination, reduced wave functions become nonorthogonal. Moreover, the reduction of wave eigenfunctions changes expectation values of all the operators except for the energy operator. It is difficult to choose the right set of eigenfunctions because they are nonorthogonal. Any mistake in choosing eigenfunctions results in complex energy modes.

In this connection, a presence of complex values in the energy spectrum of particles in the magnetic field found in Refs. [22, 23, 24, 25] is quite natural. In these works, three initial wave equations have been used. These equations have been rearranged in appropriate second-order forms. Obtained second-order wave equations are non-Hermitian. It is no wonder that corresponding energy modes has been found to be complex.

After the diagonalization, the above equations can be represented in the following general form:

$$
\begin{gathered}
E^2\phi=\left[m^2+\boldsymbol{\pi}^2-e(1+\kappa)\boldsymbol{S}\cdot\boldsymbol{H}+\frac{e(1-\kappa)}{2m^2}\boldsymbol{\pi}_\perp^2(\boldsymbol{S}\cdot\boldsymbol{H})+\zeta\right]\phi,\\
\boldsymbol{\pi}_\perp=\boldsymbol{\pi}-(\boldsymbol{\pi}\cdot\boldsymbol{e}_H)\boldsymbol{e}_H,
\end{gathered}
\qquad (35)
$$

where ζ is the designation for other terms of first and higher orders in the magnetic field strength and $\boldsymbol{e}_H=\boldsymbol{H}/H$. The quantities ζ are different for different equations (see Refs. [22, 23, 24, 25]). Eq. (35) can be transformed to the first-order form by the method proposed in Ref. [42] (see below). In the weak-field approximation,

$$
E\phi=\left[\epsilon'-\frac{e(1+\kappa)}{2\epsilon'}\boldsymbol{S}\cdot\boldsymbol{H}+\frac{e(1-\kappa)}{4m^2\epsilon'}\boldsymbol{\pi}_\perp^2(\boldsymbol{S}\cdot\boldsymbol{H})+\frac{1}{2\epsilon'}\zeta\right]\phi,\quad \epsilon'=\sqrt{m^2+\boldsymbol{\pi}^2}. \qquad (36)
$$

First-order equation (36) is consistent neither with the PKS Lagrangian [2, 3] nor with the BMT equation [27]. It follows from Eq. (36) that the angular velocity of spin precession increases when the particle energy increases. This property also contradicts the BMT equation.

Owing to difficulties discovered in some spin-1 theories, the problem of their consistency has been posed (see Ref. [1] and references therein). However, the above consideration shows this problem exists only for second-order spin-1 equations. The first-order CS equations are fully self-consistent. The problem of self-consistency of the DKP equation has been investigated in Ref. [1].

To obtain the correct second-order wave equation, the method elaborated in Ref. [42] can be used. It is based on both the Feshbach-Villars [43] and FW transformations. In Ref. [42], the connection between first-order and second-order wave equations has been found.

The general form of the second-order wave equation is given by

$$\left[\left(i\frac{\partial}{\partial t}-V\right)^2-(\boldsymbol{p}-\boldsymbol{a})^2-m^2\right]\psi=0, \tag{37}$$

where the operators V and $\boldsymbol{a}$ characterize the interaction of a particle with an external field. These operators can have arbitrary forms and involve the operators of coordinate $\boldsymbol{r}$, momentum $\boldsymbol{p}$, and spin $\boldsymbol{S}$. The wave function ψ is a spinor.

In order to linearize Eq. (37), we introduce the functions η and ζ defined by the conditions

$$\psi=\eta+\zeta, \qquad \left(i\frac{\partial}{\partial t}-V\right)\psi=m(\eta-\zeta).$$

Eq. (37) is equivalent to the following linear equation for the wave function $\Psi'=\begin{pmatrix}\eta\\ \zeta\end{pmatrix}$ (see Ref. [44]):

$$i\frac{\partial\Psi'}{\partial t}=\mathcal{H}_0\Psi'=\left[V+\rho_3\left(\frac{\boldsymbol{\pi'}^2}{2m}+m\right)+i\rho_2\frac{\boldsymbol{\pi'}^2}{2m}\right]\Psi',$$

where $\boldsymbol{\pi'}=\boldsymbol{p}-\boldsymbol{a}$ and ρ_i $(i=1,2,3)$ are the Pauli matrices.

The Hamiltonian $\mathcal{H}_0$ can be represented as

$$\mathcal{H}_0=\rho_3\mathcal{M}+V+\mathcal{O}, \quad \mathcal{M}=\frac{\boldsymbol{\pi'}^2}{2m}+m, \quad \mathcal{O}=i\rho_2\frac{\boldsymbol{\pi'}^2}{2m}, \tag{38}$$

where V and $\mathcal{O}$ are the even and odd operators, commuting and anticommuting with ρ_3, respectively. This Hamiltonian is pseudo-Hermitian.

In general $(V\neq 0)$, Hamiltonian (38) can be transformed to a block-diagonal form in two steps. Assuming that the interaction energy V is small in relation to the total energy of a relativistic particle ($|V|\ll E$), we can first make a transformation with the operator U expressed by Eq. (9). As a result, the Hamiltonian is given by Eq. (12). In this equation, the odd term $\mathcal{O}'$ is anti-Hermitian.

Since the condition $|\mathcal{O}'|\ll E$ is now satisfied, we can perform, at the second step, a transformation that is similar to the FW transformation for nonrelativistic particles (see above). If we take into account only the largest corrections, the final Hamiltonian is defined by formula (14).

We retain only terms of order V^2 and $\partial^2 V/\partial x_i\partial x_j$, disregarding terms proportional to $(\nabla V)^2$ and derivatives of V of the third order and higher. In this approximation, the transformed Hamiltonian takes the form [42]

$$\mathcal{H}''=\rho_3\epsilon'+V-\frac{1}{16}\left\{\frac{1}{\epsilon'^4},(\boldsymbol{\pi'}\cdot\nabla)(\boldsymbol{\pi'}\cdot\nabla)V\right\}_+, \tag{39}$$

where $\epsilon' = \sqrt{m^2 + \boldsymbol{\pi}'^2}$.

The inverse problem can be also solved. The first-order wave equation can be written in the form

$$\mathcal{H}'' = \rho_3\epsilon' + W, \tag{40}$$

where

$$W = V - \frac{1}{16}\left\{\frac{1}{\epsilon'^4}, (\boldsymbol{\pi}' \cdot \nabla)(\boldsymbol{\pi}' \cdot \nabla)V\right\}_+. \tag{41}$$

Therefore, the approximate form of the corresponding second-order wave equation is given by

$$\left[\left(i\frac{\partial}{\partial t} - W - \frac{1}{16}\left\{\frac{1}{\epsilon'^4}, (\boldsymbol{\pi}' \cdot \nabla)(\boldsymbol{\pi}' \cdot \nabla)W\right\}_+\right)^2 - \boldsymbol{\pi}'^2 - m^2\right]\psi = 0. \tag{42}$$

Use of Eqs. (40)–(42) makes it possible to find the second-order wave equation for relativistic spin-1 particles interacting with the electromagnetic field. This equation corresponds to first-order Eq. (20) and has the form

$$\begin{gathered}
\left[\left(i\frac{\partial}{\partial t} - V\right)^2 - \boldsymbol{\pi}^2 - m^2\right]\psi = 0,\\
V = e\Phi + \frac{e}{4m}\left[\left\{\left(\frac{g-2}{2} + \frac{m}{\epsilon'+m}\right)\frac{1}{\epsilon'}, (\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}] - \boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])\right\}_+ -\right.\\
\left.\rho_3\left\{\left(g - 2 + \frac{2m}{\epsilon'}\right), \boldsymbol{S}\cdot\boldsymbol{H}\right\}_+ + \rho_3\left\{\frac{g-2}{2\epsilon'(\epsilon'+m)}, \{\boldsymbol{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}\cdot\boldsymbol{H}\}_+\right\}_+\right] +\\
\frac{e(g-1)}{2m^2}\left(\boldsymbol{S}\cdot\nabla - \frac{1}{\epsilon'(\epsilon'+m)}(\boldsymbol{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\nabla)\right)\left(\boldsymbol{S}\cdot\boldsymbol{E} - \frac{1}{\epsilon'(\epsilon'+m)}(\boldsymbol{S}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{E})\right) +\\
\frac{e}{4m^2}\left\{\frac{1}{\epsilon'(\epsilon'+m)}\left(g - 1 + \frac{m}{\epsilon'+m}\right), \left(\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\nabla]\right)\left(\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]\right)\right\}_+ -\\
\frac{e(g-1)}{2m^2}\nabla\cdot\boldsymbol{E} + \frac{e(g-1)}{4m^2}\left\{\frac{1}{\epsilon'^2}, (\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})\right\}_+, \qquad \boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}.
\end{gathered} \tag{43}$$

Eq. (43) is Hermitian. Unfortunately, it is difficult to obtain a compact four-dimensional form of this equation.

It is evident that Eq. (43) essentially differs from the above discussed wave equations of the second order. Eq. (43) contains the three-component wave function ψ, whereas wave functions of other second-order wave equations have four components.

IX. CLASSIFICATION OF SPIN-1 PARTICLE INTERACTIONS

The results obtained in Refs. [2, 3, 5] and present work makes it possible to give the complete classification of spin-1 particle interactions. These results are expressed by Eqs.

(3),(20),(21), and (32). First of all, the best agreement between them can be pointed out. The classical and quantum theories differ only in few terms of a purely quantum origin.

In Hamilton operators (20) and (21), the first and second terms are spin-independent. They characterize the interaction of the charge e with the electromagnetic field. Lagrangian $\mathcal{L}_1$ expressed by Eq. (3) and the third terms in Hamilton operators (20) and (21) describe the electromagnetic interaction of the magnetic moment of a relativistic particle.

As it known, the preferred value of the factor g is 2 because only this value makes the quantum electrodynamics be renormalizable [35]. The general expression for the magnetic moment is given by

$$\mu = \frac{egS}{2m}. \tag{44}$$

Therefore, the preferred magnetic moment of spin-1 particles equals

$$\mu = \mu_0 = \frac{e}{m}. \tag{45}$$

In the renormalizable electroweak theory charged vector bosons $W^{\pm}$ have the magnetic moment defined by Eq. (45) [35].

For spin-1/2 and spin-1 particles, the terms in the Hamiltonians describing the electromagnetic interactions of the charge and magnetic moment are rather similar. They differ only due to different spin matrices.

Spin-1 particles possess the quadrupole moment and root-mean-square radius. These quantities are nonzero even for particles that do not have a charge distribution. For such particles, the operators of quadrupole and contact interactions are proportional to $g-1$. The electromagnetic interaction of the quadrupole moment is defined by the first term in Lagrangian $\mathcal{L}_2$ and fourth terms in Hamiltonians (20) and (21). The contact interaction caused by the root-mean-square radius is given by the sixth and seventh terms in these Hamiltonians. This interaction has not been considered in Refs. [2, 3]. The quadrupole moment and root-mean-square radius are expressed by the formulae

$$Q = -\frac{e(g-1)}{m^2}, \qquad \tau_0 = \frac{e(g-1)}{m^2}. \tag{46}$$

These formulae has been obtained in Ref. [13]. Let us note that the quadrupole operator

$$Q_{ij} = S_iS_j + S_jS_i$$

used by Young and Bludman does not include the contact part $-4\delta_{ij}/3$.

Preferred values of the quadrupole moment and root-mean-square radius for spin-1 particles are given by the substitution of $g = 2$ into Eq. (46). They are equal to

$$Q = -\frac{e}{m^2}, \qquad \tau_0 = \frac{e}{m^2}. \tag{47}$$

These values are attributed to charged vector bosons $W^{\pm}$. The preferred value of the quadrupole moment has been obtained in Refs. [2, 3, 45].

Due to AMMs of particles, the quantities μ, Q and τ_0 can be nonzero even for uncharged particles without any charge distribution. In this case, it is necessary to replace eg by $2\mu m$ in all the formulae and then put $e = 0$. Extended particles (charged and uncharged) can also possess the charge quadrupole moment and charge root-mean-square radius defined by the charge distribution.

For uncharged spin-1 particles, Eq. (46) takes the form

$$Q = -\frac{2\mu}{m}, \qquad \tau_0 = \frac{2\mu}{m}. \tag{48}$$

Formulae (46)–(48) are valid for particles possessing neither the charge quadrupole moment nor the charge root-mean-square radius. For such particles, the relativistic dependence of the quadrupole and contact interactions is given by formulae (20),(21). As follows from Eqs. (3) and (32), the relativistic dependence of the quadrupole interaction remains unchanged for particles with the charge quadrupole moment. For particles with the charge root-mean-square radius, an analogous property follows from Eq. (32).

Eqs. (3),(20),(21) also describe two interactions that do not have any classical analogues. One of these interactions (the convection interaction [2, 3]) is defined by the second term in Lagrangian $\mathcal{L}_2$ and the fifth terms in Eqs. (20),(21). The second interaction is characterized by the term

$$\frac{e}{16}\left\{\frac{1}{\epsilon'^4}, (\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})\right\}_+ .$$

This is an extra term in comparison with the classical expression for the contact interaction [see formulae (20),(21), and (32)]. This term has not been calculated in Refs. [2, 3]. Similar term enters the Hamiltonian for spin-1/2 particles (the Blount term [31, 46]). Both of the interactions vanish in the nonrelativistic limit.

X. DISCUSSION AND SUMMARY

The above analysis shows the wave equations for spin-1 particles can be verified. The FW transformation provides a good possibility of verification. This transformation can be performed for relativistic particles by the method elaborated in Refs. [28, 31]. The final Hamiltonian is block-diagonal (diagonal in two spinors).

In the present work, the Hamilton operator in the Foldy-Wouthuysen representation for relativistic spin-1 particles interacting with the nonuniform electric and uniform magnetic fields is found. The more general case of the nonuniform magnetic field is not considered because of cumbersome calculations. The performed analysis shows the full agreement between the CS equations [12], the PKS approach [2, 3], and the classical theory [5]. On the contrary, the classical equation of spin motion found by Good and Nyborg [38, 39] is unsatisfactory.

Therefore, the first-order CS equations correctly describe, at least, weak-field effects. However, the attempt of an allowance for the charge quadrupole moment fulfilled by adding appropriate second-order terms to the Lagrangian [13] does not lead to the correct result. On the contrary, the PKS approach makes it possible to find the right Lagrangian for particles of any spin possessing the charge quadrupole moment. This conclusion poses a serious problem.

The Stuckelberg equations also need an analogous investigation. To find the corresponding spin motion equation, the FW transformation can be performed for relativistic particles. This transformation has been made in the nonrelativistic case [47]. Let us mark that the BMT equation leads to the relation $g_1 - g_0 = 1/2$ between the coefficients g_0, g_1, g_2 used in Ref. [47].

In contradistinction to Refs. [2, 3], Hamiltonians (20),(21) are calculated with an allowance for spin-independent terms proportional to $\partial E_i/\partial x_j$, which describe, in particular, the contact interaction of relativistic particles. The results obtained in Refs. [2, 3, 5] and the present work make it possible to give the complete classification of spin-1 particle interactions.

Owing to difficulties discovered in some spin-1 theories, the problem of their consistency has been posed (see Ref. [1] and references therein). The present work proves this problem exists only for second-order spin-1 equations. A majority of second-order wave equations for spin-1 particles has been obtained with an elimination of several components of wave function. Derived equations are non-Hermitian. In the general case, non-Hermitian equations have complex eigenvalues and nonorthogonal eigenfunctions. The reduction of wave eigenfunctions changes expectation values of all the operators except for the energy operator. It is difficult to choose the right set of eigenfunctions because they are nonorthogonal. Any mistake in choosing eigenfunctions results in complex energy modes. In this connection, non-Hermitian wave equations are not quite satisfactory.

In the present work, the Hermitian second-order wave equation for spin-1 particles is derived by the method proposed in Ref. [42]. The found equation contains the three-component wave function, whereas wave functions of other second-order wave equations have four components.

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