Spin Oscillations in Storage Rings

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Abstract

The dependence of the particle rotation frequency on the particle orbit perturbations is found. The exact equation of spin motion in the cylindrical coordinate system is derived. The calculated formula for the frequency of g-2 precession is in the best agreement with previous results. Nevertheless, this formula contains the additional oscillatory term that can be used for fitting. The influence of spin oscillations on the spin dynamics in the EDM experiment is negligible.

The goal of this investigation is obtaining general formulae for spin oscillations in storage rings. The problem is very important because such oscillations change spin motion parameters. The spin oscillations are caused by coherent betatron oscillations (CBOs) and field distortions. In the particle rest frame, coherent betatron oscillations lead to oscillations of magnetic field acting on the particle and its spin. As a result, the spin oscillates. These oscillations result in a change of the spin rotation frequency in the g-2 experiment. If the particle possesses an electric dipole moment (EDM), they change a vertical spin motion measured in the EDM experiment. Perhaps, spin oscillations can also affect other experiments.

It is quite natural to describe the spin motion in the cylindrical coordinate system. However, the axes of this system are defined by the position of the particle which rotates and oscillates. The transformation of the Bargmann-Michel-Telegdi (BMT) [1] equation to the cylindrical coordinates should be performed with an allowance for oscillatory terms in the particle motion equation.

It is convenient to use the unit vector, $\boldsymbol{n} = \boldsymbol{p}/p$, which defines the direction of particle motion. The particle motion equation takes the form

$$\frac{d\boldsymbol{n}}{dt} = \boldsymbol{\omega} \times \boldsymbol{n}, \quad \boldsymbol{\omega} = -\frac{e}{\gamma m}\left(\boldsymbol{B} - \frac{\boldsymbol{n} \times \boldsymbol{E}}{\beta}\right),$$

where $\boldsymbol{\omega}$ is the angular velocity of the particle rotation.

Vertical and radial CBOs change the plans of particle and spin motion. The (pseudo)vectors of angular velocities become tilted. The angle Φ between two positions of any rotating vector $\boldsymbol{n}$ does not equal to the angle ϕ between two corresponding horizontal projections. Therefore, any additional field changes instantaneous frequencies of particle and spin motion. The average frequencies of particle / spin motion in the tilted and horizontal planes are the same.

The instantaneous angular velocity of particle or spin rotation in the horizontal plane is equal to

$$\dot{\phi} = \frac{(\boldsymbol{n}_{\|} \times \dot{\boldsymbol{n}}_{\|}) \cdot \boldsymbol{e}_z}{|\boldsymbol{n}_{\|}|^2} = \omega_3 - \frac{(\omega_1 n_1 + \omega_2 n_2) n_3}{1 - n_3^2}, \tag{1}$$

where $1 \Rightarrow \boldsymbol{e}_x$ or $\boldsymbol{e}_\rho$, $2 \Rightarrow \boldsymbol{e}_y$ or $\boldsymbol{e}_\phi$, $3 \Rightarrow \boldsymbol{e}_z$. This equation is exact.

Usually, we can take into account only perturbations of particle orbit caused by the vertical and radial CBOs. In this case,

$$n_1 = n_\rho = p_\rho/p = \rho_0 \sin(\omega_y t + \alpha), \quad n_3 = n_z = p_z/p = \psi_0 \sin(\omega_p t + \beta).$$

The second term in Eq. (1) is of the third order in the angular amplitudes of oscillations, ρ_0 and ψ_0. Moreover, it oscillates and therefore it equals zero on the average. If we take into account only second-order terms in the angular amplitudes, the last term in Eq. (1) is negligible. Approximately, $\dot{\phi} = \omega_3$. The horizontal axes ($\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$) rotate with the instantaneous angular velocity $\boldsymbol{\omega}' = \dot{\phi}\boldsymbol{e}_z$.

Formulae for the electric dipole moment (EDM) can be obtained from the corresponding formulae for the anomalous magnetic moment by the substitution $\boldsymbol{B}\rightarrow\boldsymbol{E},\ \boldsymbol{E}\rightarrow-\boldsymbol{B},\ g{-}2\rightarrow\eta$. The account of the particle EDM leads to the following modification of the BMT equation:

$$\begin{gathered}\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\Omega}\times\boldsymbol{s}, \quad \boldsymbol{\Omega} = \boldsymbol{\Omega}_{BMT} + \boldsymbol{\Omega}_{EDM},\\ \boldsymbol{\Omega}_{BMT} = -\frac{e}{m}\left[\left(a+\frac{1}{\gamma}\right)\boldsymbol{B} - \frac{a\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B}) - \left(a+\frac{1}{\gamma+1}\right)(\boldsymbol{\beta}\times\boldsymbol{E})\right],\\ \boldsymbol{\Omega}_{EDM} = -\frac{e\eta}{2m}\left(\boldsymbol{E} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E}) + \boldsymbol{\beta}\times\boldsymbol{B}\right), \quad a = \frac{g-2}{2},\end{gathered} \tag{2}$$

where $\boldsymbol{\Omega}$ means the angular velocity of spin rotation in the Cartesian coordinates, and $\boldsymbol{\Omega}_{BMT}$ is defined by the BMT equation. The EDM does not affect the particle motion. As a rule, we can neglect term $\frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E})$.

The spin motion equation in the cylindrical coordinate system takes the form

$$\begin{gathered}\frac{ds_\rho}{dt} = \frac{d\boldsymbol{s}}{dt}\cdot\boldsymbol{e}_\rho + \boldsymbol{s}\cdot\frac{d\boldsymbol{e}_\rho}{dt} = \boldsymbol{s}\cdot(\boldsymbol{e}_\rho\times\boldsymbol{\Omega}) + \dot{\phi}s_\phi,\\ \frac{ds_\phi}{dt} = \frac{d\boldsymbol{s}}{dt}\cdot\boldsymbol{e}_\phi + \boldsymbol{s}\cdot\frac{d\boldsymbol{e}_\phi}{dt} = \boldsymbol{s}\cdot(\boldsymbol{e}_\phi\times\boldsymbol{\Omega}) - \dot{\phi}s_\rho,\\ \frac{ds_z}{dt} = \boldsymbol{s}\cdot(\boldsymbol{e}_z\times\boldsymbol{\Omega}).\end{gathered} \tag{3}$$

We can introduce the coordinate system with the axes $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ corresponding to the axes $\boldsymbol{e}_\rho, \boldsymbol{e}_\phi, \boldsymbol{e}_z$, respectively. For such a system, Eq. (3) can be rewritten in the form:

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\omega}_a\times\boldsymbol{s}, \quad \boldsymbol{\omega}_a = \boldsymbol{\Omega} - \dot{\phi}\boldsymbol{e}_3 \tag{4}$$

or

$$\begin{gathered}\boldsymbol{\omega}_a = -\frac{e}{m}\left\{a\boldsymbol{B} - \frac{a\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B}) + \left(\frac{1}{\gamma^2-1} - a\right)(\boldsymbol{\beta}\times\boldsymbol{E}) + \frac{1}{\gamma}[\boldsymbol{B}_{\|}\right.\\ \left.-\frac{1}{\beta^2}(\boldsymbol{\beta}\times\boldsymbol{E})_{\|}] + \frac{\eta}{2}\left(\boldsymbol{E} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E}) + \boldsymbol{\beta}\times\boldsymbol{B}\right)\right\} + \frac{(\omega_1 n_1 + \omega_2 n_2)n_3}{1-n_3^2}\boldsymbol{e}_3,\end{gathered} \tag{5}$$

where the sign $\|$ means the components parallel to the x_1x_2-plane. Eqs. (4),(5) are exact, and ω_a is the angular frequency of g-2 precession. These equations describe the spin motion in an arbitrary storage ring with an allowance for the particle and spin oscillations and the EDM. As a rule, the last term in Eq. (5) is negligible.

In the g-2 experiment, the influence of the vertical CBO (pitch) on the spin rotation frequency has been calculated by Farley [2, 3]. The result has been confirmed by Field and Fiorentini [4] and computer simulations.

The horizontal CBO (yaw) does not give any significant corrections.

With the above formulae, the theory of spin oscillations in the g-2 experiment can be developed in the very general form. The spin motion perturbed by the vertical CBO is described by the equation

$$\begin{aligned} \frac{d\boldsymbol{s}}{dt} &= \{a_0 + a_3\cos\left[2(\omega_p t + \phi_p)\right]\}(\boldsymbol{e}_3\times\boldsymbol{s}) \\ &+ a_2\cos(\omega_p t + \phi_p)(\boldsymbol{e}_2\times\boldsymbol{s}) + a_1\sin(\omega_p t + \phi_p)(\boldsymbol{e}_1\times\boldsymbol{s}), \end{aligned} \tag{6}$$

where a_1 and a_2 are first-order quantities and a_3 is a second-order quantity in the angular pitch amplitude. The quantity ω_p is the angular pitch frequency.

This equation has the exact solution. Averaged g-2 frequency equals

$$\begin{aligned} \omega_a = \omega_0(1-C), \qquad C &= \frac{1}{4}\psi_0^2\left[1 - \frac{\omega_0^2}{\gamma^2(\omega_0^2-\omega_p^2)} - \frac{\omega_p^2(f-1)(f-1+2/\gamma)}{\omega_0^2-\omega_p^2}\right. \\ &\left. - \frac{(\gamma-1)^2\omega_0^2 - f^2\gamma^2\omega_p^2}{\gamma^2(\omega_0^2-\omega_p^2)} < \cos\left[2(\omega_0 t + \phi_0)\right] > \cdot \frac{1+{s_3^{(0)}}^2}{1-{s_3^{(0)}}^2}\right], \end{aligned} \tag{7}$$

where

$$f = 1 + a\gamma - \frac{1+a}{\gamma} = 1 + a\beta^2\gamma - \frac{1}{\gamma} \qquad \text{and} \qquad f = 1 + a\gamma$$

for electric and magnetic focusing, respectively.

The average value of the last term oscillating with the frequency $2\omega_0$ is zero. Therefore, formula (7) is in the best agreement with previous results [2, 3, 4] found for the particular case $s_3^{(0)} = 0$. However, it is possible to include the oscillatory term in a fitting process instead of its elimination. In the current g-2 experiment, $f = 1,\ s_3^{(0)} = 0,\ \gamma \gg 1,\ \phi_0$ equals 0 or π, and formula (7) takes the form

$$\omega_a = \omega_0(1-C), \quad C = \frac{1}{4}\psi_0^2\left[1 - < \cos(2\omega_0 t) >\right].$$

This formula shows the inclusion of the oscillatory term in a fitting process is possible.

In the EDM experiment, the spin motion in the horizontal plane is strongly restricted with the radial electric field. In this case, the spin rotation about the radial axis becomes very important because this rotation imitates the EDM effect.

The vertical CBO leads to the spin motion described by the equation

$$\frac{d\boldsymbol{s}}{dt} = \{a_0 + a_3\cos{(\omega_p t + \phi_p)}\}\,(\boldsymbol{e}_3\times\boldsymbol{s}) + a_1\sin{(\omega_p t+\phi_p)}(\boldsymbol{e}_1\times\boldsymbol{s}),$$

where $1 \Rightarrow \boldsymbol{e}_\phi,\ 2 \Rightarrow \boldsymbol{e}_z,\ 3 \Rightarrow \boldsymbol{e}_\rho$, and a_1 and a_3 are first-order quantities in the angular amplitude of pitch. This equation also has the exact solution. Averaged angular frequency is given by

$$\omega_a = a_0 + \frac{a_0 a_1^2}{4(a_0^2-\omega_p^2)}\left[1 + <\cos{[2(a_0 t+\phi_0)]}> \cdot \frac{1+{s_3^{(0)}}^2}{1-{s_3^{(0)}}^2}\right] + \frac{a_1 a_3}{2\omega_p}\cdot\frac{s_3^{(0)}}{s_{\|}^{(0)}} <\sin{(a_0 t+\phi_0)}> .$$

In the EDM experiment

$$\omega_a = \omega_0\left(1 - \frac{a^2\omega_c^2}{2\omega_p^2}\psi_0^2\right),$$

where ω_c is the cyclotron frequency.

The yaw correction for the EDM experiment is nonzero:

$$\omega_a = \omega_0\left[1 + a(\gamma-1)\gamma\left(\frac{\omega_p^2}{2\omega_y^2} - \frac{1}{g}\right)\rho_0^2\right].$$

However, vertical and radial CBOs give only small corrections to the systematical errors caused by the vertical electric field. Since these systematical errors need to be eliminated, the corrections for the vertical and radial CBOs are negligible.

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