Polarization of spin-1 particles without an anomalous magnetic moment in a uniform magnetic field

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Abstract

The polarization operator projections onto four directions remain unchanged for spin-1 particles without an anomalous magnetic moment in a uniform magnetic field. The approximate conservation of the polarization operator projections onto the horizontal axes of the cylindrical coordinate system takes place.

Keywords: spin precession, spin-1 particles, uniform magnetic field

It is well-known that the projections of the polarization operator onto the directions both of the constant uniform magnetic field, $\boldsymbol{B}$, and of the kinetic momentum, $\boldsymbol{\pi}=\boldsymbol{p}-e\boldsymbol{A}$ (where $\boldsymbol{p}\equiv -i\nabla$ is the momentum operator and $\boldsymbol{A}$ is the vector potential of the magnetic field), remain unchanged for Dirac spin-1/2 particles [1, 2]. Such particles do not possess an anomalous magnetic moment (AMM) and their g factor is $g=2$. It has been found in Ref. [3] that the projection of the polarization operator onto the direction defined by the operator $\boldsymbol{\pi}\times\boldsymbol{B}$ is also constant for the Dirac spin-1/2 particles.

In the present work, we prove that similar properties are valid for spin-1 particles without the AMM. These properties are exact. We use the designations $[\ldots,\ldots]$ and $\{\ldots,\ldots\}$ for commutators and anticommutators, respectively, and the system of units $\hbar=c=1$.

The magnetic moment of any spinning particle can be split into "normal" and anomalous parts:

$$\mu=\mu_0+\mu', \quad \mu_0=\frac{eS}{m}, \quad \mu'=\mu-\mu_0.$$

One can introduce the g factor as follow:

$$g=\frac{2m\mu}{eS}.$$

As is known, the preferred g factor is equal to 2 for point-like particles [4]. Such particles do not possess the AMM. An example of a point-like spin-1 particle is the W boson. While the deuteron is a nucleus, its g factor is also close to 2 ($g_d=1.714$).

The general Hamiltonian in the Sakata-Taketani (ST) representation [5] for spin-1 particles interacting with an electromagnetic field has been obtained by Young and Bludman [6]. However, the use of the ST representation does not give a possibility to obtain a clear semiclassical limit of the relativistic quantum mechanics. To find such a limit, one should use the Foldy-Wouthuysen (FW) representation. Properties of this representation are unique (see Refs. [7, 8, 9, 10] and references therein). The Hamiltonian and all operators are block-diagonal (diagonal in two spinors). Relations between the operators in the FW representation are similar to those between the respective classical quantities. For relativistic particles in external fields, operators have the same form as in the nonrelativistic quantum theory. As a result, the FW representation provides the best possibility of obtaining a meaningful semiclassical limit of the relativistic quantum mechanics.

The wave functions of spin-1 particles are pseudo-orthogonal, e.g., their normalization is defined by the relation

$$\int \Psi^\dagger \rho_3 \Psi dV = \int (\phi^\dagger \phi - \chi^\dagger \chi) dV = 1,$$

where $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ is the six-component wave function (bispinor).

It is convenient to split six-component matrices on the spin matrices S_x, S_y, S_z and the Pauli matrices those components act on the upper and lower spinors:

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Hamiltonian for spin-1 particles is pseudo-Hermitian, that is, it satisfies the conditions

$$\mathcal{H} = \rho_3 \mathcal{H}^\dagger \rho_3, \quad \mathcal{H}^\dagger = \rho_3 \mathcal{H} \rho_3.$$

Even (diagonal) terms of the Hamiltonian are Hermitian and odd (off-diagonal) terms are anti-Hermitian.

The operator U transforming the wave function to another representation should be pseudo-unitary:

$$U^{-1} = \rho_3 U^\dagger \rho_3, \quad U^\dagger = \rho_3 U^{-1} \rho_3.$$

The general ST Hamiltonian derived in Ref. [6] is given by

$$\begin{aligned} \mathcal{H} = e\Phi + \rho_3 m + i\rho_2 \frac{1}{m}(\boldsymbol{S}\cdot\boldsymbol{D})^2 \\ -(\rho_3 + i\rho_2)\frac{1}{2m}(\boldsymbol{D}^2 + e\boldsymbol{S}\cdot\boldsymbol{B}) - (\rho_3 - i\rho_2)\frac{e\kappa}{2m}(\boldsymbol{S}\cdot\boldsymbol{B}) \\ -\frac{e\kappa}{2m^2}(1+\rho_1)\Big[(\boldsymbol{S}\cdot\boldsymbol{E})(\boldsymbol{S}\cdot\boldsymbol{D}) - i\boldsymbol{S}\cdot[\boldsymbol{E}\times\boldsymbol{D}] - \boldsymbol{E}\cdot\boldsymbol{D}\Big] \\ +\frac{e\kappa}{2m^2}(1-\rho_1)\Big[(\boldsymbol{S}\cdot\boldsymbol{D})(\boldsymbol{S}\cdot\boldsymbol{E}) - i\boldsymbol{S}\cdot[\boldsymbol{D}\times\boldsymbol{E}] - \boldsymbol{D}\cdot\boldsymbol{E}\Big], \end{aligned} \tag{1}$$

where $\boldsymbol{S}$ is the spin matrix, $\boldsymbol{B}$ is the magnetic field induction, $\kappa = \text{const}$, and $\boldsymbol{D} = \nabla - ie\boldsymbol{A}$. For spin-1 particles, the polarization operator is equal to $\boldsymbol{\Pi} = \rho_3 \boldsymbol{S}$.

For the considered case of the particle in the uniform magnetic field, Eq. (1) reduces to

$$\begin{aligned} \mathcal{H} = \rho_3 \left[m + \frac{\boldsymbol{\pi}^2}{2m} - \frac{e(\kappa+1)}{2m}\boldsymbol{S}\cdot\boldsymbol{B}\right] \\ +i\rho_2 \left[\frac{\boldsymbol{\pi}^2}{2m} - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{S})^2}{m} + \frac{e(\kappa-1)}{2m}\boldsymbol{S}\cdot\boldsymbol{B}\right], \end{aligned} \tag{2}$$

where $\boldsymbol{\pi} = -i\boldsymbol{D} = -i\nabla - e\boldsymbol{A}$ is the kinetic momentum operator.

The above Hamiltonian can be presented in the form

$$\mathcal{H}=\rho_3\mathcal{M}+\mathcal{O},\quad \rho_3\mathcal{O}=-\mathcal{O}\rho_3, \tag{3}$$

$$\begin{aligned}\mathcal{M}&=m+\frac{\boldsymbol{\pi}^2}{2m}-\frac{e(\kappa+1)}{2m}\boldsymbol{S}\cdot\boldsymbol{B},\\ \mathcal{O}&=i\rho_2\left[\frac{\boldsymbol{\pi}^2}{2m}-\frac{(\boldsymbol{\pi}\cdot\boldsymbol{S})^2}{m}+\frac{e(\kappa-1)}{2m}\boldsymbol{S}\cdot\boldsymbol{B}\right],\end{aligned} \tag{4}$$

where $\mathcal{O}$ is the odd operator anticommuting with ρ_3. The connection between the factors κ and g is given by $g=\kappa+1$ [6].

The operators $\mathcal{M}$ and $\mathcal{O}$ commute only when $\kappa=1$ $(g=2)$. The relation $[\mathcal{M},\mathcal{O}]=0$ is the sufficient condition of the exact FW transformation [10]. Therefore, the value $g=2$ is prominent not only in the field theory but also in the quantum mechanics of spin-1 particles. In this case, the exact FW Hamiltonian is equal to [10]

$$\mathcal{H}_{FW}=\rho_3\sqrt{\mathcal{M}^2+\mathcal{O}^2}$$

or

$$\mathcal{H}_{FW}=\rho_3\sqrt{m^2+\boldsymbol{\pi}^2-2e\boldsymbol{S}\cdot\boldsymbol{B}}. \tag{5}$$

This equation is similar to the corresponding one for Dirac particles $(g=2)$ [11]:

$$\mathcal{H}_{FW}=\rho_3\sqrt{m^2+\boldsymbol{\pi}^2-e\boldsymbol{\sigma}\cdot\boldsymbol{B}}, \tag{6}$$

where $\boldsymbol{\sigma}$ is the Pauli matrix. The similarity between Eqs. (5) and (6) results in analogous properties of spin-1/2 and spin-1 particles interacting with the uniform magnetic field.

Hamiltonian (5) commutes with the operator $\boldsymbol{\Pi}\cdot\boldsymbol{B}$. As $\boldsymbol{\Pi}\cdot\boldsymbol{B}=\Pi_B B$, $\boldsymbol{B}=\text{const}$, the commutator of the operators $\mathcal{H}_{FW}$ and Π_B equals zero: $[\mathcal{H}_{FW},\Pi_B]=0$. This ensures retaining the polarization operator projection onto the magnetic field direction Π_B. The relations

$$[\boldsymbol{\pi}^2,\boldsymbol{\pi}]=2ie\boldsymbol{B}\times\boldsymbol{\pi},\quad [\boldsymbol{S}\cdot\boldsymbol{B},\boldsymbol{S}\cdot\boldsymbol{\pi}]=i\boldsymbol{S}\cdot(\boldsymbol{B}\times\boldsymbol{\pi})$$

result in the commutation of the FW Hamiltonian with the operator $\boldsymbol{\Pi}\cdot\boldsymbol{\pi}$. Since $\boldsymbol{\Pi}\cdot\boldsymbol{\pi}=\Pi_l|\boldsymbol{\pi}|$ and $[\mathcal{H}_{FW},|\boldsymbol{\pi}|]\equiv[\mathcal{H}_{FW},\sqrt{\boldsymbol{\pi}^2}]=0$, so

$$[\mathcal{H}_{FW},\Pi_l]=0, \tag{7}$$

where the operator Π_l defines the longitudinal projection of the polarization operator. Therefore, this projection also remains unchanged.

The longitudinal direction is defined by the operator $\boldsymbol{\pi}$. However, the unit operator fixing this direction cannot be given by

$$\boldsymbol{l}=\frac{1}{2}\left(\boldsymbol{\pi}\frac{1}{|\boldsymbol{\pi}|}+\frac{1}{|\boldsymbol{\pi}|}\boldsymbol{\pi}\right), \tag{8}$$

because $\boldsymbol{l}^2\equiv\boldsymbol{l}\cdot\boldsymbol{l}\neq 1$. The square of the operator $\boldsymbol{l}$ is equal to

$$\begin{aligned}\boldsymbol{l}^2&=\frac{1}{4}\left(1+\left\{\frac{1}{|\boldsymbol{\pi}|},\boldsymbol{\pi}\frac{1}{|\boldsymbol{\pi}|}\boldsymbol{\pi}\right\}+\boldsymbol{\pi}\frac{1}{\boldsymbol{\pi}^2}\boldsymbol{\pi}\right)\\&=1-\frac{1}{8}\left(\left\{\frac{1}{|\boldsymbol{\pi}|},\left[\boldsymbol{\pi},\left[\boldsymbol{\pi},\frac{1}{|\boldsymbol{\pi}|}\right]\right]\right\}+\left[\boldsymbol{\pi},\left[\boldsymbol{\pi},\frac{1}{\boldsymbol{\pi}^2}\right]\right]\right).\end{aligned}$$

The double commutators can be calculated with Eqs. (24),(25) from Ref. [9]. When we suppose the parameter $|e|B/\boldsymbol{\pi}^2$ to be small with respect to 1, the approximate expression for the square of the operator $\boldsymbol{l}$ is

$$\boldsymbol{l}^2=1+\frac{3e^2B^2}{4\boldsymbol{\pi}^4}-\frac{7e^2(\boldsymbol{\pi}\cdot\boldsymbol{B})^2}{4\boldsymbol{\pi}^6}. \tag{9}$$

The commutation of the operators $\mathcal{H}_{FW}$ and $\boldsymbol{\Pi}\cdot(\boldsymbol{\pi}\times\boldsymbol{B})$ is proved in a similar way. As

$$\begin{aligned}\left[\boldsymbol{\pi}^2,(\boldsymbol{\Pi}\cdot[\boldsymbol{\pi}\times\boldsymbol{B}])\right]&=[\boldsymbol{\pi}^2,\boldsymbol{\pi}]\cdot[\boldsymbol{B}\times\boldsymbol{\Pi}]\\&=2ie[\boldsymbol{B}\times\boldsymbol{\pi}]\cdot[\boldsymbol{B}\times\boldsymbol{\Pi}],\\ [(\boldsymbol{S}\cdot\boldsymbol{B}),(\boldsymbol{\Pi}\cdot[\boldsymbol{\pi}\times\boldsymbol{B}])]&=\rho_3\left[(\boldsymbol{S}\cdot\boldsymbol{B}),(\boldsymbol{S}\cdot[\boldsymbol{\pi}\times\boldsymbol{B}])\right]\\&=i\rho_3\boldsymbol{S}\cdot[\boldsymbol{B}\times[\boldsymbol{\pi}\times\boldsymbol{B}]]=i[\boldsymbol{B}\times\boldsymbol{\pi}]\cdot[\boldsymbol{B}\times\boldsymbol{\Pi}],\end{aligned}$$

the operator under the radical sign in Eq. (5) commutes with $\boldsymbol{\Pi}\cdot(\boldsymbol{\pi}\times\boldsymbol{B})$, and, hence, the Hamiltonian $\mathcal{H}_{FW}$ also commutes with this operator: $[\mathcal{H}_{FW},(\boldsymbol{\Pi}\cdot[\boldsymbol{\pi}\times\boldsymbol{B}])]=0$.

It is easy to prove [3] that the Hamilton operator commutes with

$$|\boldsymbol{\pi}\times\boldsymbol{B}|\equiv\sqrt{(\boldsymbol{\pi}\times\boldsymbol{B})^2}=\sqrt{\boldsymbol{\pi}^2B^2-(\boldsymbol{\pi}\cdot\boldsymbol{B})^2}.$$

As $[\pi_i,\pi_j]=iee_{ijk}B_k$ (e_{ijk} is the antisymmetric unit pseudotensor), so the operator $\boldsymbol{\pi}\cdot\boldsymbol{B}$ commutes with any projection of the kinetic momentum operator $\boldsymbol{\pi}$. Since

$$[\mathcal{H},\boldsymbol{\pi}^2]=0,\quad [\boldsymbol{\pi}^2,(\boldsymbol{\pi}\cdot\boldsymbol{B})^2]=\left\{[\boldsymbol{\pi}^2,(\boldsymbol{\pi}\cdot\boldsymbol{B})],(\boldsymbol{\pi}\cdot\boldsymbol{B})\right\}=0,$$

the operator under the radical sign in Eq. (5) commutes with $|\boldsymbol{\pi}\times\boldsymbol{B}|$. Whence it follows that $[\mathcal{H}_{FW},|\boldsymbol{\pi}\times\boldsymbol{B}|]=0$ [3]. As $\boldsymbol{\Pi}\cdot(\boldsymbol{\pi}\times\boldsymbol{B})=\Pi_t|\boldsymbol{\pi}\times\boldsymbol{B}|$, the polarization operator projection onto the transversal direction $\boldsymbol{\pi}\times\boldsymbol{B}$ commutes with the Hamiltonian:

$$[\mathcal{H}_{FW},\Pi_t]=0. \tag{10}$$

As a result, the transversal projection of the polarization operator also remains unchanged at the motion of the particle.

We can similarly prove that

$$[\mathcal{H}_{FW}, (\mathbf{\Pi}\cdot[\boldsymbol{B}\times(\boldsymbol{\pi}\times\boldsymbol{B})])] = 0,$$
$$[\mathcal{H}_{FW}, |\boldsymbol{B}\times(\boldsymbol{\pi}\times\boldsymbol{B})|] = 0.$$

These relations result in the conservation of the polarization operator projection onto the direction $\boldsymbol{B}\times(\boldsymbol{\pi}\times\boldsymbol{B})$:

$$[\mathcal{H}_{FW}, \Pi_{B\pi B}] = 0, \tag{11}$$

where $\Pi_{B\pi B}$ is defined by $\mathbf{\Pi}\cdot[\boldsymbol{B}\times(\boldsymbol{\pi}\times\boldsymbol{B})] = \Pi_{B\pi B}|\boldsymbol{B}\times(\boldsymbol{\pi}\times\boldsymbol{B})|$. The same property is also valid for spin-1/2 particles.

The problem of conservation of the polarization operator projections onto the axes of the cylindrical coordinate system is rather important [3]. If the z axis in parallel to the field direction ($\boldsymbol{B} = B\boldsymbol{e}_z$), then the polarization operator projections onto the directions of vectors $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$ have the form

$$\Pi_\rho = \mathbf{\Pi}\cdot\boldsymbol{e}_\rho = \Pi_x\cos\phi + \Pi_y\sin\phi,$$
$$\Pi_\phi = \mathbf{\Pi}\cdot\boldsymbol{e}_\phi = -\Pi_x\sin\phi + \Pi_y\cos\phi.$$

These projections would be conserved at condition that

$$[\mathcal{H}, \Pi_\rho] = 0, \quad [\mathcal{H}, \Pi_\phi] = 0, \tag{12}$$

or

$$[\mathcal{F}, \Pi_\rho] = 0, \quad [\mathcal{F}, \Pi_\phi] = 0, \quad \mathcal{F} = m^2 + \boldsymbol{\pi}^2 - 2e\boldsymbol{S}\cdot\boldsymbol{B}. \tag{13}$$

Since

$$\boldsymbol{S}\cdot\boldsymbol{B} = S_zB, \quad [S_z, \Pi_\rho] = i\Pi_\phi, \quad [S_z, \Pi_\phi] = -i\Pi_\rho,$$

the commutators in Eq. (13) are equal to

$$\begin{aligned}
[\mathcal{F}, \Pi_\rho] &= -i\left(\left\{\pi_\phi, \frac{\Pi_\phi}{\rho}\right\} + 2eB\Pi_\phi\right),\\
[\mathcal{F}, \Pi_\phi] &= i\left(\left\{\pi_\phi, \frac{\Pi_\rho}{\rho}\right\} + 2eB\Pi_\rho\right),\\
\pi_\phi &= \frac{1}{2}\Bigl(-\{\pi_x, \sin\phi\} + \{\pi_y, \cos\phi\}\Bigr).
\end{aligned} \tag{14}$$

Formulae (14) which are similar to the corresponding equations for spin-1/2 particles [3] show that conditions (12),(13) cannot be satisfied in the general case. However, these conditions are approximately satisfied when a particle motion can be semiclassically described. Semiclassical description is possible, if the orbital angular moment of particles is large enough:

$$L = r|P_\phi| \gg 1,$$

where r is the radius of the circular orbit, and P_ϕ is the projection of the classical momentum of particles. In this case, the radius r is defined by

$$r = -\frac{P_\phi}{eB}$$

and the commutators of the operator π_ϕ with the coordinate operators are negligible (see Refs. [2, 3, 12]).

The operators π_ϕ and ρ are defined on the class of functions that are the eigenfunctions of $\mathcal{H}$. With allowance for the semiclassical nature of the motion and the possibility to ignore the noncommutativity of the operator π_ϕ with the coordinate operators, we can therefore replace the operators π_ϕ and ρ by the values P_ϕ and r, respectively. Thus, conditions (12),(13) are approximately satisfied.

Therefore, the polarization operator projections onto the radial and azimuthal directions of the axes of the cylindrical coordinate system, Π_ρ and Π_ϕ, are approximately conserved, when the orbital angular moment of particles is large with respect to 1. The spin-1/2 particles possess the same property [3].

It is proved that the polarization operator projections onto the vertical, longitudinal, transversal directions and the direction orthogonal to the vertical and transversal ones are conserved for spin-1 particles without the AMM in a uniform magnetic field. Spin-1/2 particles without the AMM possess the same property. The conservation of the polarization operator projections onto the horizontal axes of the cylindrical coordinate system is approximate.

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