Foldy-Wouthuysen Transformation and Semiclassical Transition for Relativistic Quantum Mechanics

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Abstract

It is shown that the Foldy-Wouthuysen transformation for relativistic particles in strong external fields provides the possibility of obtaining a meaningful classical limit of the relativistic quantum mechanics. The full agreement between quantum and classical theories is proved. The coincidence of the semiclassical equations of motion of particles and their spins with the corresponding classical equations is established. The Niels Bohr's correspondence principle is valid not only in the limit of large spin quantum numbers but also for particles with any spin as well as for spinless particles.

Keywords: Foldy-Wouthuysen transformation; relativistic quantum mechanics; semiclassical transition; correspondence principle

I. INTRODUCTION

The Foldy-Wouthuysen (FW) transformation has been proposed in Ref. [1]. Its main goals are *i*) transformation of Dirac Hamiltonians to the block-diagonal (diagonal in two spinors) form and *ii*) establishment of connection between the relativistic quantum mechanics and the classical physics. We utilize the method of the FW transformation for relativistic particles in strong external fields developed in Ref. [2]. This method uses the expansion of the FW Hamiltonian into a power series in the Planck constant which characterizes the order of magnitude of quantum corrections. We present the FW Hamiltonians, quantum mechanical and semiclassical equations of motion of scalar, spin-1/2, and spin-1 particles and their spins in electromagnetic fields and similar equations for spin-1/2 particles in gravitational fields and noninertial frames.

We also discuss the extension of the correspondence principle resulting from the presented semiclassical equations.

The system of units $\hbar = c = 1$ is used.

II. METHODS OF FOLDY-WOUTHUYSEN TRANSFORMATION

The Foldy-Wouthuysen (FW) representation occupies a special place in relativistic quantum mechanics thank to its unique properties. This representation provides the best possibility of obtaining a meaningful classical limit of the relativistic quantum mechanics (see Refs. [2, 3] and references therein).

The advantages of the FW transformation can be formulated as follows. Relations between the operators in the FW representantion are similar to those between the respective classical quantities. For relativistic particles in external fields, operators have the same form as in the nonrelativistic quantum theory. For example, the position operator is $\boldsymbol{r}$ and the momentum one is $\boldsymbol{p} = -i\hbar\nabla$. The transition to the semiclassical description is very simple and consists in trivial replacing operators by corresponding classical quantities. For relativistic particles, the connection between the square of the wave function and the probability of a definite position of a particle is restored: $w(\boldsymbol{r}) = |\Psi(\boldsymbol{r})|^2$.

Initial Hamiltonian for spin-1/2 particles is

$$\mathcal{H} = \beta m + \mathcal{E} + \mathcal{O}, \quad \beta\mathcal{E} = \mathcal{E}\beta, \quad \beta\mathcal{O} = -\mathcal{O}\beta. \tag{1}$$

Original method by Foldy and Wouthuysen [1] transforms it to the form

$$\mathcal{H}_{FW}=\beta\left(m+\frac{\mathcal{O}^2}{2m}-\frac{\mathcal{O}^4}{8m^3}\right)+\mathcal{E}-\frac{1}{8m^2}\left[\mathcal{O},\left[\mathcal{O},\mathcal{E}\right]\right]-\frac{i}{8m^2}\left[\mathcal{O},\dot{\mathcal{O}}\right]. \tag{2}$$

This is a nonrelativistic transformation with relativistic corrections. The FW transformation for relativistic spin-1/2 particles results in [3]

$$\begin{gathered}
\mathcal{H}_{FW}=\beta\epsilon+\mathcal{E}'+\frac{\beta}{4}\left\{\mathcal{O}'^2,\frac{1}{\epsilon}\right\}, \quad \epsilon=\sqrt{m^2+\mathcal{O}^2},\\
\mathcal{E}'=\mathcal{E}-\frac{1}{4}\left[\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}},\left[\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}},\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\right]\right]\\
-\frac{1}{4}\left[\frac{\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}},\left[\frac{\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}},\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\right]\right],\\
\mathcal{O}'=\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}-\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}.
\end{gathered} \tag{3}$$

The FW transformation for relativistic particles with arbitrary spin in strong external fields has been investigated in Ref. [2]. The FW Hamiltonian can be expanded into the power series in the Planck constant, $\hbar$. Initial equation is

$$\mathcal{H}=\beta\mathcal{M}+\mathcal{E}+\mathcal{O}, \quad \beta\mathcal{M}=\mathcal{M}\beta, \quad \beta\mathcal{E}=\mathcal{E}\beta, \quad \beta\mathcal{O}=-\mathcal{O}\beta. \tag{4}$$

The operators $\mathcal{M}$ and $\mathcal{E}$ are even while the operator $\mathcal{O}$ is odd.

In Ref. [2], the method developed in Ref. [3] has been generalized in order to take into account a possible non-commutativity of the operators $\mathcal{M}$ and $\mathcal{O}$. First transformation is performed with the transformation operator U defined by

$$U=\frac{\beta\epsilon+\beta\mathcal{M}-\mathcal{O}}{\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}}\beta, \quad U^{-1}=\beta\frac{\beta\epsilon+\beta\mathcal{M}-\mathcal{O}}{\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}}, \quad \epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}^2}, \tag{5}$$

where $U^{-1}=U^\dagger$ when $\mathcal{H}=\mathcal{H}^\dagger$ and $U^{-1}=U^\ddagger$ when $\mathcal{H}=\mathcal{H}^\ddagger$. The latter case takes place for spinless and spin-1 particles. The sign "‡" denotes the pseudo-Hermitian conjugate and means $\mathcal{H}^\ddagger\equiv\beta\mathcal{H}^\dagger\beta$. For particles with any spin, $\beta\equiv\sigma_3\otimes I$, where σ_3 is the 2×2 Pauli matrix and I is the corresponding unit matrix. The used form of the transformation operator allows to perform the FW transformation in the general case.

We consider the general case when external fields are nonstationary. The exact formula for the transformed Hamiltonian has the form

$$\begin{aligned}\mathcal{H}' = \beta\epsilon + \mathcal{E} + \frac{1}{2T}\Biggl([T,[T,(\beta\epsilon+\mathcal{F})]] + \beta\,[\mathcal{O},[\mathcal{O},\mathcal{M}]]\\ - [\mathcal{O},[\mathcal{O},\mathcal{F}]] - [(\epsilon+\mathcal{M}),[(\epsilon+\mathcal{M}),\mathcal{F}]] - [(\epsilon+\mathcal{M}),[\mathcal{M},\mathcal{O}]]\\ -\beta\,\{\mathcal{O},[(\epsilon+\mathcal{M}),\mathcal{F}]\} + \beta\,\{(\epsilon+\mathcal{M}),[\mathcal{O},\mathcal{F}]\}\Biggr)\frac{1}{T},\end{aligned} \tag{6}$$

where $\mathcal{F} = \mathcal{E} - i\hbar\frac{\partial}{\partial t}$ and $T = \sqrt{(\beta\epsilon + \beta\mathcal{M} - \mathcal{O})^2}$.

Hamiltonian (6) still contains odd terms proportional to the first and higher powers of the Planck constant. This Hamiltonian can be presented in the form

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}' + \mathcal{O}', \quad \beta\mathcal{E}' = \mathcal{E}'\beta, \quad \beta\mathcal{O}' = -\mathcal{O}'\beta, \tag{7}$$

where $\epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2}$. The even and odd parts of Hamiltonian (7) are defined by the well-known relations:

$$\mathcal{E}' = \frac{1}{2}\left(\mathcal{H}' + \beta\mathcal{H}'\beta\right) - \beta\epsilon, \quad \mathcal{O}' = \frac{1}{2}\left(\mathcal{H}' - \beta\mathcal{H}'\beta\right).$$

Additional transformations performed according to Refs. [2, 3] bring $\mathcal{H}'$ to the block-diagonal form. The approximate formula for the final FW Hamiltonian is [2]

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}' + \frac{1}{4}\beta\left\{\mathcal{O}'^2, \frac{1}{\epsilon}\right\}. \tag{8}$$

Eqs. (6),(8) solve the problem of the FW transformation for relativistic particles of arbitrary spin in strong external fields.

Eq. (6) can be significantly simplified in some special cases. When $[\mathcal{M},\mathcal{O}] = 0$ and the external fields are stationary, it is reduced to

$$\begin{aligned}\mathcal{H}' = \beta\epsilon + \mathcal{E} + \frac{1}{2T}\Biggl([T,[T,\mathcal{E}]]\\ - [\mathcal{O},[\mathcal{O},\mathcal{E}]] - [(\epsilon+\mathcal{M}),[(\epsilon+\mathcal{M}),\mathcal{E}]]\\ -\beta\,\{\mathcal{O},[(\epsilon+\mathcal{M}),\mathcal{E}]\} + \beta\,\{(\epsilon+\mathcal{M}),[\mathcal{O},\mathcal{E}]\}\Biggr)\frac{1}{T}.\end{aligned} \tag{9}$$

In this case, $[\epsilon,\mathcal{M}] = [\epsilon,\mathcal{O}] = 0$ and the operator $T = \sqrt{2\epsilon(\epsilon+\mathcal{M})}$ is even.

The FW transformations becomes exact, when $[\mathcal{M},\mathcal{O}] = 0$, $[\mathcal{O},\mathcal{E}] = 0$, and the external fields are stationary. In this case [2]

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}, \quad \epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2}. \tag{10}$$

The exact FW transformation can be performed in the general case by the Eriksen method [4]. The validity of the Eriksen transformation has been argued by de Vries and Jonker [5]. The Eriksen transformation operator has the form [4]

$$U=\frac{1}{2}(1+\beta\lambda)\left[1+\frac{1}{4}(\beta\lambda+\lambda\beta-2)\right]^{-1/2}, \quad \lambda=\frac{\mathcal{H}}{(\mathcal{H}^2)^{1/2}}, \tag{11}$$

where $\mathcal{H}$ is the Hamiltonian in the Dirac representation. This operator brings the Dirac wave function and the Dirac Hamiltonian to the FW representation in one step. However, it is difficult to use the Eriksen method for obtaining an explicit form of the relativistic FW Hamiltonian because the general final formula is very cumbersome and contains roots of Dirac matrix operators. Therefore, the Eriksen method was not used for relativistic particles in external fields.

Other methods of the FW transformation have been developed in Refs. [6, 7, 8, 9].

III. QUANTUM MECHANICAL AND SEMICLASSICAL EQUATIONS OF MOTION OF PARTICLES AND THEIR SPINS

A. Equations of Motion in Electromagnetic Fields

FW Hamiltonians have been derived for relativistic scalar particles [2, 10], relativistic spin-1/2 particles with electric and magnetic dipole moments [2] in strong electromagnetic fields, and relativistic spin-1 particles without electric dipole moments (EDMs) [11] in a strong uniform magnetic field.

In the FW representation, the transition to the semiclassical approximation becomes trivial. It consists in replacing operators by corresponding classical quantities. The quantum mechanical equations of motion of particles and their spins are given by

$$\frac{d\boldsymbol{\pi}}{dt}=\frac{i}{\hbar}[\mathcal{H}_{FW},\boldsymbol{\pi}]-\frac{e}{c}\cdot\frac{\partial\boldsymbol{A}}{\partial t}, \quad \boldsymbol{\pi}=\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}, \tag{12}$$

$$\frac{d\boldsymbol{\Pi}}{dt}=\frac{i}{\hbar}[\mathcal{H}_{FW},\boldsymbol{\Pi}], \tag{13}$$

where $\boldsymbol{\pi}$ is the kinetic momentum operator and $\boldsymbol{\Pi}$ is the polarization operator. Usual definitions of the Dirac matrices are applied.

The equation of spin-1/2 particle motion in the strong electromagnetic field to within first-order terms in the Planck constant has the form [2]

$$
\begin{aligned}
\frac{d\boldsymbol{\pi}}{dt} = e\boldsymbol{E} + \beta\frac{ec}{4}\left\{\frac{1}{\epsilon'}, ([\boldsymbol{\pi}\times\boldsymbol{B}] - [\boldsymbol{B}\times\boldsymbol{\pi}])\right\} + \mu'\nabla(\boldsymbol{\Pi}\cdot\boldsymbol{B}) + \frac{\mu_0}{2}\left\{\frac{mc^2}{\epsilon'}, \nabla(\boldsymbol{\Pi}\cdot\boldsymbol{H})\right\} \\
-\frac{\mu' c}{4}\left\{\frac{1}{\epsilon'}, [\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]) - \nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])]\right\} \\
-\frac{\mu_0 mc^3}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}[\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]) - \nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])]\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}} \\
-\frac{\mu' c^2}{2\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Pi}\cdot\boldsymbol{\pi}), [\nabla(\boldsymbol{H}\cdot\boldsymbol{\pi}) + \nabla(\boldsymbol{\pi}\cdot\boldsymbol{B})]\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}.
\end{aligned}
\tag{14}
$$

This equation can be divided into two parts. The first part does not contain the Planck constant and describes the quantum equivalent of the Lorentz force. The second part is of order of $\hbar$. This part defines the relativistic expression for the Stern-Gerlach force. Small terms proportional to d are omitted.

The equation of spin motion is given by [2]

$$
\begin{aligned}
\frac{d\boldsymbol{\Pi}}{dt} = \frac{2\mu'}{\hbar}\boldsymbol{\Sigma}\times\boldsymbol{B} + \frac{\mu_0}{\hbar}\left\{\frac{mc^2}{\epsilon'}, \boldsymbol{\Sigma}\times\boldsymbol{B}\right\} - \frac{\mu' c}{2\hbar}\left\{\frac{1}{\epsilon'}, [\boldsymbol{\Pi}\times(\boldsymbol{\pi}\times\boldsymbol{E}) - \boldsymbol{\Pi}\times(\boldsymbol{E}\times\boldsymbol{\pi})]\right\} \\
-\frac{\mu_0 mc^3}{\hbar\sqrt{\epsilon'(\epsilon'+mc^2)}}[\boldsymbol{\Pi}\times(\boldsymbol{\pi}\times\boldsymbol{E}) - \boldsymbol{\Pi}\times(\boldsymbol{E}\times\boldsymbol{\pi})]\frac{1}{\sqrt{\epsilon'(\epsilon'+mc^2)}} \\
-\frac{\mu' c^2}{\hbar\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Sigma}\times\boldsymbol{\pi}), (\boldsymbol{H}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{B})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}} \\
+\frac{2d}{\hbar}\boldsymbol{\Sigma}\times\boldsymbol{E} - \frac{dc^2}{\hbar\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Sigma}\times\boldsymbol{\pi}), (\boldsymbol{E}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{E})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}} \\
+\frac{dc}{2\hbar}\left\{\frac{1}{\epsilon'}, [\boldsymbol{\Pi}\times(\boldsymbol{\pi}\times\boldsymbol{B}) - \boldsymbol{\Pi}\times(\boldsymbol{B}\times\boldsymbol{\pi})]\right\}.
\end{aligned}
\tag{15}
$$

Eqs. (14),(15) describe strong-field effects.

For spinless particles, the operator equation of particle motion takes the form [2]

$$
\frac{d\boldsymbol{\pi}}{dt} = e\boldsymbol{E} + \beta\frac{ec}{4}\left\{\frac{1}{\epsilon}, ([\boldsymbol{\pi}\times\boldsymbol{B}] - [\boldsymbol{B}\times\boldsymbol{\pi}])\right\}. \tag{16}
$$

The right hand side of this equation coincides with the spin-independent part of the corresponding equation for spin-1/2 particles. Additional terms in the operator equation of particle motion derived in Ref. [10] are of order of $\hbar^2$.

Similar equations have been derived for spin-1 particles in a strong magnetic field. When the matrices $\boldsymbol{\Sigma} = I\boldsymbol{S}$ ($\boldsymbol{S}$ is the 3×3 spin matrix) and $\boldsymbol{\Pi} = \rho_3\boldsymbol{S}$ acting on the functions ϕ and χ are entered, the operator equations of motion of spin-1 particles and their spins are given by [11]

$$
\begin{gathered}
\frac{d\boldsymbol{\pi}}{dt}=-\rho_3\frac{ec}{2}\left\{\frac{1}{\epsilon'},\boldsymbol{B}\times\boldsymbol{\pi}\right\}-\frac{e^2\hbar c^2}{2}(\boldsymbol{\Pi}\cdot\boldsymbol{B})\left\{\frac{1}{\epsilon'^3},\boldsymbol{B}\times\boldsymbol{\pi}\right\} \\
+\frac{e^2\hbar(g-2)}{2m}(\boldsymbol{B}\cdot\boldsymbol{\pi})\left[\frac{\boldsymbol{\Pi}\times\boldsymbol{B}}{\epsilon'(\epsilon'+mc^2)}\right. \\
\left.+\frac{c^2}{4}\left\{\frac{2\epsilon'+mc^2}{\epsilon'^3(\epsilon'+mc^2)^2},\{(\boldsymbol{B}\times\boldsymbol{\pi}),(\boldsymbol{\Pi}\cdot\boldsymbol{\pi})\}\right\}\right],
\end{gathered}
\tag{17}
$$

$$
\begin{gathered}
\frac{d\boldsymbol{\Pi}}{dt}=\left[\frac{e(g-2)}{2mc}+\frac{ec}{\epsilon'}\right]\boldsymbol{\Sigma}\times\boldsymbol{B} \\
-\frac{ec(g-2)}{4m}(\boldsymbol{B}\cdot\boldsymbol{\pi})\left\{\frac{1}{\epsilon'(\epsilon'+mc^2)},\boldsymbol{\Sigma}\times\boldsymbol{\pi}\right\}.
\end{gathered}
\tag{18}
$$

Eqs. (17) and (18) are derived with allowance for terms up to the first and zero orders in the Planck constant, respectively. As a result, Eq. (17) describes the quantum equivalent of the Lorentz force and defines the relativistic expression for the Stern-Gerlach force. The Stern-Gerlach force is rather weak in the uniform magnetic field because it is of order of B^2.

The semiclassical limit of the relativistic quantum mechanics has been investigated in Ref. [2]. The expansion into a power series in the Planck constant can be available only if

$$pl\gg\hbar, \tag{19}$$

where p is the momentum of the particle and l is the characteristic size of the nonuniformity region of the external field. This relation is equivalent to

$$\lambda\ll l, \tag{20}$$

where λ is the de Broglie wavelength. Eqs. (19),(20) result from the fact that the Planck constant appears in the final Hamiltonian due to commutators between the operators $\mathcal{M},\mathcal{E}$, and $\mathcal{O}$.

One needs to average the operators in the quantum mechanical equations. When the FW representation is used and relations (19),(20) are valid, the semiclassical transition consists in trivial replacing operators with corresponding classical quantities. If the momentum and position operators are chosen to be the dynamical variables, relations (19),(20) are equivalent to the condition

$$|<p_i>|\cdot|<x_i>|\gg|<[p_i,x_i]>|=\hbar,\quad i=1,2,3. \tag{21}$$

The angular brackets which designate averaging in time will be hereinafter omitted.

As a result of replacing operators by corresponding classical quantities, the semiclassical equations of motion of spin-1/2 particles and their spins take the form

$$\begin{gathered}
\frac{d\boldsymbol{\pi}}{dt}=e\boldsymbol{E}+\frac{ec}{\epsilon'}\left(\boldsymbol{\pi}\times\boldsymbol{H}\right)+\mu'\nabla(\boldsymbol{P}\cdot\boldsymbol{H})+\frac{\mu_0}{mc^2\epsilon'}\nabla(\boldsymbol{P}\cdot\boldsymbol{H})\\
-\frac{\mu' c}{\epsilon'}\nabla(\boldsymbol{P}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}])-\frac{\mu_0 mc^3}{\epsilon'(\epsilon'+mc^2)}\nabla(\boldsymbol{P}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}])\\
-\frac{\mu' c^2}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\cdot\boldsymbol{\pi})\nabla(\boldsymbol{H}\cdot\boldsymbol{\pi}),\qquad \boldsymbol{P}=\frac{\boldsymbol{s}}{S},
\end{gathered} \tag{22}$$

$$\begin{gathered}
\frac{d\boldsymbol{P}}{dt}=2\mu'\boldsymbol{P}\times\boldsymbol{H}+\frac{2\mu_0 mc^2}{\epsilon'}(\boldsymbol{P}\times\boldsymbol{H})-\frac{2\mu' c}{\epsilon'}\left(\boldsymbol{P}\times[\boldsymbol{\pi}\times\boldsymbol{E}]\right)\\
-\frac{2\mu_0 mc^3}{\epsilon'(\epsilon'+mc^2)}\left(\boldsymbol{P}\times[\boldsymbol{\pi}\times\boldsymbol{E}]\right)-\frac{2\mu' c^2}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\times\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{H})\\
+2d\boldsymbol{P}\times\boldsymbol{E}-\frac{2dc^2}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\times\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{E})+\frac{2dc}{\epsilon'}\left(\boldsymbol{P}\times[\boldsymbol{\pi}\times\boldsymbol{H}]\right).
\end{gathered} \tag{23}$$

In Eqs. (22),(23), $\epsilon'=\sqrt{m^2c^4+c^2\boldsymbol{\pi}^2}$, $\boldsymbol{P}$ is the polarization vector, $\boldsymbol{s}$ is the spin vector (i.e., the average spin), and S is the spin quantum number.

Similar semiclassical equation of spin motion for spin-1 particles has been derived in Ref. [11].

For scalar particles

$$\frac{d\boldsymbol{\pi}}{dt}=e\boldsymbol{E}+\frac{ec}{\sqrt{m^2c^4+c^2\boldsymbol{\pi}^2}}\left(\boldsymbol{\pi}\times\boldsymbol{H}\right). \tag{24}$$

Two first terms in right hand sides of Eqs. (22),(24) are the same as in the classical expression for the Lorentz force. This is a manifestation of the Niels Bohr's correspondence principle. The part of Eq. (23) dependent on the magnetic moment coincides with the well-known Thomas-Bargmann-Michel-Telegdi (T-BMT) equation. It is natural because the T-BMT equation has been derived without the assumption that the external fields are weak. The whole Eq. (23) coincides with the corresponding classical equation derived in Ref. [12]. The relativistic formula for the Stern-Gerlach force can be obtained from the Lagrangian consistent with the T-BMT equation (see Ref. [13]). The semiclassical and classical formulas describing this force also coincide. High-order corrections in $\hbar$ to the quantum equations of motion of particles and their spins bring a difference between quantum and classical approaches.

B. Equations of Motion in Gravitational Fields and Noninertial Frames

The best compliance between the description of spin effects in the classical and quantum gravity has been proved in Refs. [14, 15]. In these works, some Hamiltonians in the Dirac representation derived in Refs. [16, 17] from the initial covariant Dirac equation have been used. The initial Dirac Hamiltonians have been transformed to the FW representation by the method elaborated in Ref. [3].

The exact transformation of the Dirac equation for the metric

$$ds^2 = V^2(\boldsymbol{r})(dx^0)^2 - W^2(\boldsymbol{r})(d\boldsymbol{r}\cdot\boldsymbol{r}) \tag{25}$$

to the Hamiltonian form has been carried out by Obukhov [16]:

$$i\frac{\partial\psi}{\partial t} = \mathcal{H}\psi, \quad \mathcal{H} = \beta m V + \frac{1}{2}\{\mathcal{F}, \boldsymbol{\alpha}\cdot\boldsymbol{p}\}, \tag{26}$$

where $\mathcal{F} = V/W$. Hamiltonian (26) covers several cases including a weak Schwarzschild field in the isotropic coordinates and a uniformly accelerated frame.

The relativistic FW Hamiltonian derived in Ref. [14] has the form

$$\begin{gathered}
\mathcal{H}_{FW} = \beta\epsilon + \frac{\beta}{2}\left\{\frac{m^2}{\epsilon}, V-1\right\} + \frac{\beta}{2}\left\{\frac{\boldsymbol{p}^2}{\epsilon}, \mathcal{F}-1\right\} \\
-\frac{\beta m}{4\epsilon(\epsilon+m)}\Big[\boldsymbol{\Sigma}\cdot(\boldsymbol{\phi}\times\boldsymbol{p}) - \boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{\phi}) + \nabla\cdot\boldsymbol{\phi}\Big] \\
+\frac{\beta m(2\epsilon^3+2\epsilon^2 m+2\epsilon m^2+m^3)}{8\epsilon^5(\epsilon+m)^2}(\boldsymbol{p}\cdot\nabla)(\boldsymbol{p}\cdot\boldsymbol{\phi}) \\
+\frac{\beta}{4\epsilon}\left[\boldsymbol{\Sigma}\cdot(\boldsymbol{f}\times\boldsymbol{p}) - \boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{f}) + \nabla\cdot\boldsymbol{f}\right] - \frac{\beta(\epsilon^2+m^2)}{4\epsilon^5}(\boldsymbol{p}\cdot\nabla)(\boldsymbol{p}\cdot\boldsymbol{f}),
\end{gathered} \tag{27}$$

where $\epsilon = \sqrt{m^2+\boldsymbol{p}^2}$, $\boldsymbol{\phi} = \nabla V$, $\boldsymbol{f} = \nabla\mathcal{F}$.

The operator equations of momentum and spin motion take the form [14]

$$\begin{gathered}
\frac{d\boldsymbol{p}}{dt} = i[\mathcal{H}_{FW}, \boldsymbol{p}] = -\frac{\beta}{2}\left\{\frac{m^2}{\epsilon}, \boldsymbol{\phi}\right\} - \frac{\beta}{2}\left\{\frac{\boldsymbol{p}^2}{\epsilon}, \boldsymbol{f}\right\} \\
+\frac{m}{2\epsilon(\epsilon+m)}\nabla(\boldsymbol{\Pi}\cdot(\boldsymbol{\phi}\times\boldsymbol{p})) - \frac{1}{2\epsilon}\nabla(\boldsymbol{\Pi}\cdot(\boldsymbol{f}\times\boldsymbol{p}))
\end{gathered} \tag{28}$$

and

$$\frac{d\boldsymbol{\Pi}}{dt} = \frac{m}{\epsilon(\epsilon+m)}\boldsymbol{\Sigma}\times(\boldsymbol{\phi}\times\boldsymbol{p}) - \frac{1}{\epsilon}\boldsymbol{\Sigma}\times(\boldsymbol{f}\times\boldsymbol{p}), \tag{29}$$

respectively.

The semiclassical equations of motion are [14]

$$\frac{d\boldsymbol{p}}{dt} = -\frac{m^2}{\epsilon}\boldsymbol{\phi} - \frac{\boldsymbol{p}^2}{\epsilon}\boldsymbol{f} + \frac{m}{2\epsilon(\epsilon+m)}\nabla(\boldsymbol{P}\cdot(\boldsymbol{\phi}\times\boldsymbol{p})) - \frac{1}{2\epsilon}\nabla(\boldsymbol{P}\cdot(\boldsymbol{f}\times\boldsymbol{p})) \tag{30}$$

and

$$\frac{d\boldsymbol{P}}{dt} = \frac{m}{\epsilon(\epsilon+m)}\boldsymbol{P}\times(\boldsymbol{\phi}\times\boldsymbol{p}) - \frac{1}{\epsilon}\boldsymbol{P}\times(\boldsymbol{f}\times\boldsymbol{p})\,, \tag{31}$$

respectively. In Eq. (30), two latter terms describe the force dependent on the spin. This force is similar to the electromagnetic Stern-Gerlach force and is rather weak. The angular velocity of spin rotation is given by

$$\boldsymbol{\Omega} = -\frac{m}{\epsilon(\epsilon+m)}(\boldsymbol{\phi}\times\boldsymbol{p}) + \frac{1}{\epsilon}(\boldsymbol{f}\times\boldsymbol{p})\,. \tag{32}$$

We can find similar equations describing a change of the direction of particle momentum, $\boldsymbol{n} = \boldsymbol{p}/p$:

$$\frac{d\boldsymbol{n}}{dt} = \boldsymbol{\omega}\times\boldsymbol{n}, \quad \boldsymbol{\omega} = \frac{m^2}{\epsilon p}(\boldsymbol{\phi}\times\boldsymbol{n}) + \frac{p}{\epsilon}(\boldsymbol{f}\times\boldsymbol{n}). \tag{33}$$

Explicit form of the equation of motion of the three-component spin has been obtained by Pomeransky and Khriplovich [13]. The derivation of this equation is based on neglecting the relatively weak influence of the spin on a particle's trajectory which results in a weak violation of the equivalence principle by the curvature-dependent terms [18]. In this approximation, the Pomeransky-Khriplovich equations (PKEs) for the three-component and four-component spins agree with the seminal Mathisson-Papapetrou equations for the four-component spin [19] (see Ref. [20] and references therein).

A simple calculation shows [14] that Eqs. (30)–(33) coincide with the corresponding classical equations of motion of particles and their spins obtained from the PKEs for given metric (25). The gravitational analogue of the Stern-Gerlach force defined by Eq. (30) coincides with the corresponding force obtained from the PKEs for the three-component spin (see Ref. [20]).

The FW Hamiltonian and the operators of velocity and acceleration have also been calculated for the Dirac particle in the rotating frame [15]. The exact Dirac Hamiltonian derived in Ref. [17] has been used. In Ref. [15], perfect agreement between classical and quantum approaches has also been established. The operators of velocity and acceleration are equal to

$$\boldsymbol{v} = \beta\frac{\boldsymbol{p}}{\epsilon} - \boldsymbol{\omega}\times\boldsymbol{r}, \quad \epsilon = \sqrt{m^2+\boldsymbol{p}^2},$$
$$\boldsymbol{w} = 2\beta\frac{\boldsymbol{p}\times\boldsymbol{\omega}}{\epsilon} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = 2\boldsymbol{v}\times\boldsymbol{\omega} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}). \tag{34}$$

Quantum mechanical formula (34) for the acceleration of the relativistic spin-1/2 particle coincides with the classical formula [21] for the sum of the Coriolis and centrifugal accelerations. Obtained results also agree with the corresponding nonrelativistic formulas from [17].

Quantum equations of motion of Dirac particles and their spins in a gravitational field of a rotating body defined by the Lense-Thirring metric being a weak field limit of the Kerr metric have been derived in Ref. [22]. The equation of rotation of the spin contains two parts. One of them is defined by the static part of the Lense-Thirring (LT) metric and is expressed by Eqs. (29) and (31). The second part is proportional to the total angular momentum of the source, $\boldsymbol{J} = Mca\boldsymbol{e}_z$, and is defined by the operator of angular velocity of the spin precession [22]:

$$\begin{aligned}\boldsymbol{\Omega}_{LT} \;=\; & \frac{G}{c^2r^3}\left[\frac{3(\boldsymbol{r}\cdot\boldsymbol{J})\boldsymbol{r}}{r^2}-\boldsymbol{J}\right]-\frac{3G}{4}\left\{\frac{1}{\epsilon(\epsilon+mc^2)},\left[\frac{2\{\boldsymbol{l},(\boldsymbol{J}\cdot\boldsymbol{l})\}}{r^5}\right.\right.\\ &\left.\left.+\frac{1}{2}\left\{(\boldsymbol{p}\times\boldsymbol{l}-\boldsymbol{l}\times\boldsymbol{p}),\frac{(\boldsymbol{r}\cdot\boldsymbol{J})}{r^5}\right\}+\left\{(\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{J})),\frac{1}{r^3}\right\}\right]\right\}. \qquad (35)\end{aligned}$$

The semiclassical formula corresponding to Eq. (35) and describing the motion of average spin has the form [22]

$$\boldsymbol{\Omega}_{LT}=\frac{G}{c^2r^3}\left[\frac{3(\boldsymbol{r}\cdot\boldsymbol{J})\boldsymbol{r}}{r^2}-\boldsymbol{J}\right]-\frac{3G}{r^3\epsilon(\epsilon+mc^2)}\left[\frac{2\boldsymbol{l}(\boldsymbol{J}\cdot\boldsymbol{l})+(\boldsymbol{p}\times\boldsymbol{l})(\boldsymbol{r}\cdot\boldsymbol{J})}{r^2}+\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{J})\right]. \qquad (36)$$

This equation can also be expressed in the equivalent form [22]:

$$\boldsymbol{\Omega}^{(2)}=\frac{G}{c^2r^3}\left[\frac{3(\boldsymbol{r}\cdot\boldsymbol{J})\boldsymbol{r}}{r^2}-\boldsymbol{J}\right]-\frac{3G}{r^5\epsilon(\epsilon+mc^2)}\left[\boldsymbol{l}(\boldsymbol{l}\cdot\boldsymbol{J})+(\boldsymbol{r}\cdot\boldsymbol{p})(\boldsymbol{p}\times(\boldsymbol{r}\times\boldsymbol{J}))\right]. \qquad (37)$$

The equation of motion of the particle defines the evolution of the contravariant four-momentum operator which spatial components $(a, b = 1, 2, 3)$ are given by

$$p^a = g^{ab}p_b + g^{0a}p_0.$$

In a stationary metric, the evolution of the contravariant momentum operator in the weak field approximation is defined by

$$F^a=\frac{dp^a}{dt}=-\frac{dp_a}{dt}+\frac{1}{4}\left\{\left\{v^b,\frac{\partial g^{ai}}{\partial x^b}\right\},p_i\right\}, \qquad \frac{d\boldsymbol{p}}{dt}=\frac{i}{\hbar}[\mathcal{H}_{FW},\boldsymbol{p}], \qquad (38)$$

where F^a is the force operator and $v^a \approx \beta c^2p^a/\epsilon \approx c^2p^a/\mathcal{H}_{FW}$ is the velocity operator.

The force operator caused by the LT effect is equal to [22]

$$\boldsymbol{F}=\frac{c}{2}\left(\operatorname{curl}\boldsymbol{K}\times\boldsymbol{p}-\boldsymbol{p}\times\operatorname{curl}\boldsymbol{K}\right)+\boldsymbol{F}_s, \tag{39}$$

where

$$\operatorname{curl}\boldsymbol{K}=\frac{2G}{c^3r^3}\left[\frac{3(\boldsymbol{r}\cdot\boldsymbol{J})\boldsymbol{r}}{r^2}-\boldsymbol{J}\right],\quad \boldsymbol{F}_s=-\nabla\left(\frac{\hbar G}{2c^2r^3}\left[\frac{3(\boldsymbol{r}\cdot\boldsymbol{J})(\boldsymbol{r}\cdot\boldsymbol{\Sigma})}{r^2}-\boldsymbol{J}\cdot\boldsymbol{\Sigma}\right]\right.$$
$$-\frac{3\hbar G}{8}\left\{\frac{1}{\epsilon(\epsilon+mc^2)},\left[\frac{2\{(\boldsymbol{J}\cdot\boldsymbol{l}),(\boldsymbol{\Sigma}\cdot\boldsymbol{l})\}}{r^5}+\frac{1}{2}\left\{\left(\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{l})-\boldsymbol{\Sigma}\cdot(\boldsymbol{l}\times\boldsymbol{p})\right),\frac{(\boldsymbol{r}\cdot\boldsymbol{J})}{r^5}\right\}\right.\right.$$
$$\left.\left.\left.+\left\{\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{J})),\frac{1}{r^3}\right\}\right]\right\}\right). \tag{40}$$

These equations are given without allowance for contributions from V, W. The part of operator equations (39) and (40) defining the spin-independent force is in the best compliance with the corresponding classical equation [23]. Since the Dirac spin operator is $\boldsymbol{s}=\hbar\boldsymbol{\Sigma}/2$, Eqs. (39) and (40) yield the corresponding semiclassical equation [22]:

$$\boldsymbol{\mathcal{F}}=c\operatorname{curl}\boldsymbol{K}\times\boldsymbol{p}+\boldsymbol{\mathcal{F}}_s, \tag{41}$$

$$\boldsymbol{\mathcal{F}}_s=-\nabla\left(\frac{G}{c^2r^3}\left[\frac{3(\boldsymbol{r}\cdot\boldsymbol{J})(\boldsymbol{r}\cdot\boldsymbol{s})}{r^2}-\boldsymbol{J}\cdot\boldsymbol{s}\right]\right.$$
$$\left.-\frac{3G}{\epsilon(\epsilon+mc^2)}\left[\frac{2(\boldsymbol{J}\cdot\boldsymbol{l})(\boldsymbol{s}\cdot\boldsymbol{l})}{r^5}+\frac{(\boldsymbol{s}\cdot[\boldsymbol{p}\times\boldsymbol{l}])\,(\boldsymbol{r}\cdot\boldsymbol{J})}{r^5}+\frac{(\boldsymbol{s}\cdot[\boldsymbol{p}\times[\boldsymbol{p}\times\boldsymbol{J}]])}{r^3}\right]\right). \tag{42}$$

The relativistic result (40), (42) for the spin-dependent force perfectly agrees with the corresponding nonrelativistic classical formulas previously obtained in Ref. [24] on the basis of the Mathisson-Papapetrou equations [19].

The presented quantum equations agree with the classical results obtained with the PKEs. This follows from the fact that the spin-dependent part of the Hamiltonian has the form $\mathcal{H}_s=\hbar(\boldsymbol{\Omega}^{(1)}\cdot\boldsymbol{\Sigma}+\boldsymbol{\Omega}^{(2)}\cdot\boldsymbol{\Pi})/2$ that perfectly agrees with the general classical Eq. (47) of Ref. [13].

Thus, the classical and quantum approaches are in full agreement. Purely quantum effects are not too important. They consist in appearing some additional terms in the FW Hamiltonian. However, the leading corrections are proportional to derivatives of $\boldsymbol{\phi}$ and $\boldsymbol{f}$ and are similar to the well-known Darwin term in the electrodynamics. As a result, the influence of the additional terms on the motion of particles and their spins in gravitational fields can be neglected. In this case, the classical and semiclassical equations of motion of particles and their spins coincide.

IV. FOLDY-WOUTHUYSEN TRANSFORMATION AND NIELS BOHR'S CORRESPONDENCE PRINCIPLE

The correspondence principle has been formulated by Niels Bohr [25]. This principle is very important because it establishes a connection between classical and quantum physics. The correspondence principle states that the behavior of quantum mechanical systems reproduces the classical physics in the limit of large quantum numbers [25]. As follows from this statement, the quantum mechanics should generate classical results in the limit of $S \to \infty$ only.

The classical limit of the relativistic quantum mechanics can be obtained with averaging operators (i.e., substituting classical quantities for corresponding operators), neglecting commutators, and vanishing the Planck constant. For particles with any spins, the problem can be unambiguously solved with the FW transformation because operators in the FW representation have the same form as in the nonrelativistic quantum theory.

When high-order terms in $\hbar$ are omitted, the semiclassical equations of motions of particles and their spins coincide with the corresponding classical equations. Since particles with spin 0, 1/2, and 1 are considered, one can conclude that **the behavior of quantum mechanical systems reproduces the classical physics in the limit of large quantum numbers for particles with arbitrary spin.** Thus, there is no need for the additional restriction $S \to \infty$. The Niels Bohr's correspondence principle is valid for particles with any spin as well as for spinless particles. This is an evident extension of the correspondence principle because the definitions of the spin in the initial quantum mechanical equations and classical equations significantly differ. The spin is defined by vector sums of spin matrices in the quantum mechanics and the intrinsic angular momentum in the classical physics. The form of the spin matrices depends on the spin quantum number S. So, the obtained extension of the Niels Bohr's correspondence principle is rather nontrivial.

V. DISCUSSION AND SUMMARY

We can conclude that the FW transformation for relativistic particles in strong external fields gives the meaningful classical limit of the relativistic quantum mechanics. We have established that the semiclassical equations of motion of particles and their spins coincide with the corresponding classical equations. The Niels Bohr's correspondence principle is valid not only in the limit of large spin quantum numbers but also for particles with any spin as well as for spinless particles.

To confirm the consistency of classical and quantum equations, we can additionally use the results obtained by Pomeransky and Khriplovich [13]. In this work, the quantum Lagrangians of interaction of arbitrary spin particles with electromagnetic and gravitational fields have been derived by the method of scattering amplitudes. The comparison of these Lagrangians with the corresponding classical ones shows the coincidence of terms of the zeroth and first orders in spin.

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