Validation of the Eriksen Method of the Exact FoldyWouthuysen Representation

A.J. Silenko

Research Institute for Nuclear Problems,

Belarusian State University, Minsk, 220030 Belarus

Abstract

The Eriksen method is proven to yield a correct and exact result when a sufficient condition of exact transformation to the Foldy-Wouthuysen (FW) representation is satisfied. Therefore, the Eriksen method is confirmed as valid. This makes it possible to establish the limits within which the approximate step-by-step methods are applicable. The latter is done by comparing the relativistic formulas for a Hamiltonian operator in the FW representation (obtained using those methods) and the known expression for the first terms of a series, which defines the expansion of this operator in powers of v/c as found by applying the Eriksen method.

PACS numbers: 03.65.Pm, 11.10.Ef

The transition to the FW representation (FW transformation) carried out in well-known paper [1] is a unitary transformation leading to an even Hamiltonian operator that characterizes the FW representation. A Hamiltonian operator is block-diagonal in this representation; i.e., it is diagonal in two spinors or their analogues for particles with spins $S \neq 1/2$. The FW representation has unique properties that render it special in quantum mechanics. In this representation the quantum mechanical operators for relativistic particles in an external field have the same form as in the nonrelativistic quantum theory. The relationships between operators are similar to those between respective classical quantities. The specified properties of the FW representation make it impossible for ambiguities to emerge when this representation is used to go over to the quasiclassical approximation and classical limit of relativistic quantum mechanics [1, 2].

Nonetheless, it was noticed a relatively long time ago that the block diagonalization of the Hamiltonian is not at all identical to the transition to FW representation (see Ref. [3] and references therein). Moreover, strictly speaking, even the method developed by Foldy and Wouthuysen [1] does not lead to this representation [4]. It is a step-by-step method. When using this kind of method, a transition to the block-diagonal form of the Hamiltonian is achieved by successive iterations, resulting in the elimination of odd (nondiagonal) highest order terms at each step. However, the operator of the exact FW transformation U_{FW} ($\Psi_{FW} = U_{FW}\Psi$) for spin-1/2 particles must satisfy the Eriksen condition [5]:

$$\beta U_{FW} = U_{FW}^\dagger \beta, \tag{1}$$

where β is a Dirac matrix. When the operator U_{FW} is represented in the exponential form

$$U_{FW} = \exp{(iS)} \tag{2}$$

condition (1) is equivalent to requiring that the exponential operator S is Hermitian and odd [4]. By virtue of the Hausdorff theorem [6], the step-by-step methods are shown [4] to not satisfy the S operator oddness condition. Therefore, they can ensure only an approximate transition to the FW representation.

An original Hamiltonian for spin-1/2 particles can be represented in the following general form:

$$\mathcal{H}_D = \beta m + \mathcal{E} + \mathcal{O}, \quad \beta\mathcal{E} = \mathcal{E}\beta, \quad \beta\mathcal{O} = -\mathcal{O}\beta, \tag{3}$$

where $\mathcal{E}$ and $\mathcal{O}$ are even and odd operators, respectively.

The general form of the operator of exact FW representation was found by Eriksen [5]:

$$U_{FW}=\frac{1}{2}(1+\beta\lambda)\left[1+\frac{1}{4}(\beta\lambda+\lambda\beta-2)\right]^{-1/2},\quad \lambda=\frac{\mathcal{H}_D}{\sqrt{\mathcal{H}_D^2}}, \tag{4}$$

where $\lambda=+1$ and -1 for solutions with positive and negative energies, respectively. It is important that [5]

$$\lambda^2=1,\quad [\beta\lambda,\lambda\beta]=0 \tag{5}$$

and the operator $\beta\lambda+\lambda\beta$ is even:

$$[\beta,(\beta\lambda+\lambda\beta)]=0. \tag{6}$$

The even operators are block diagonal and do not mix upper and lower spinors. Formula (4) can also be represented as follows [3]:

$$U_{FW}=\frac{1+\beta\lambda}{\sqrt{(1+\beta\lambda)^\dagger(1+\beta\lambda)}}. \tag{7}$$

Two operator factors in the denominator commute.

The validity of the Eriksen transformation was argued in [7]. The operator U_{FW} turns to zero either at lower or upper spinors of any eigenfunction of the Dirac Hamiltonian. This transformation is carried out in just a single step. However, it is problematic to effectively use the Eriksen method to find relativistic formulas for particles in an external field, since the general formula (4) is rather cumbersome and contains square roots of Dirac matrices. An expression for the Hamiltonian operator in FW representation was found [7] only in the form of a relativistic correction series in powers of $\mathcal{E}/m,\ \mathcal{O}/m$. Thus, the Eriksen method does not solve the problem of finding compact relativistic expressions for the Hamiltonian operator in this representation.

Some of the step-by-step methods give those relativistic expressions [8–15]. Note that the method developed in [11] is applicable to the case of particles with arbitrary spin. Since all the methods mentioned so far are approximate, it is necessary to determine the limits of their applicability. It is clear that the simplest and most reliable way to make such a determination is to compare relativistic Hamiltonians in the FW representation obtained by using the step-by-step methods with exact expansion into a series presented in [7].

However, it is first needed to make a maximally possible verification of the Eriksen (4) or equivalent (7) formulas because a justification of the oddness of the exponent operator

S, which is part of formula (2) and is given in [4], is not rigorous mathematical proof. The fact that the Eriksen formula gives a correct answer in the case of free particles [5] is also not enough to prove its validity.

Back in 2003, a sufficient condition for carrying out an exact FW transformation was found in Ref. [10]. This transformation is exact if an external field is stationary and the operators $\mathcal{E}$ and $\mathcal{O}$ commute:

$$[\mathcal{E}, \mathcal{O}] = 0. \tag{8}$$

In this case

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}, \quad \epsilon = \sqrt{m^2 + \mathcal{O}^2}. \tag{9}$$

In the present paper we find out if the Eriksen formula is compatible with this exact transformation.

It should be indicated that, when finding a square root of operators, an obvious condition $(\sqrt{A})^2 = A$ should be supplemented by the condition of equality between a square root of unit matrix and the unit matrix itself [10]. Thus, for free particles $\mathcal{E} = 0$, $\mathcal{O} = \boldsymbol{\alpha} \cdot \boldsymbol{p}$ and $\lambda = (\beta m + \alpha \cdot \boldsymbol{p})/\sqrt{m^2 + \boldsymbol{p}^2}$.

With this in mind and considering that the operators $\mathcal{E}$ and $\mathcal{O}$ commute, a square root can be brought to the form

$$\sqrt{\mathcal{H}_D^2} = \epsilon + \frac{(\beta m + \mathcal{O})\mathcal{E}}{\epsilon} = \epsilon \left[1 + \frac{(\beta m + \mathcal{O})\mathcal{E}}{\epsilon^2}\right]. \tag{10}$$

Since the following equality

$$\mathcal{H}_D = (\beta m + \mathcal{O}) \left[1 + \frac{(\beta m + \mathcal{O})\mathcal{E}}{\epsilon^2}\right],$$

is correct, the sign operator λ assumes the form

$$\lambda = \frac{\beta m + \mathcal{O}}{\epsilon}. \tag{11}$$

It is very important that in the case at hand it is independent of $\mathcal{E}$.

The operator of exact FW transformation is given by

$$U_{FW} = \frac{\epsilon + m + \beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon + m)}}. \tag{12}$$

Formula (12) has the same form as a respective one found in [10]. It is natural that the expression for the Hamiltonian operator in FW representation obtained using the Eriksen method coincides with the result deduced in [10] provided that condition (8) is satisfied:

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}. \tag{13}$$

Thus, in the case under consideration, the Eriksen method gives a correct and exact result. It is natural that, under the additional condition (3), the original Dirac Hamiltonian (8) has a much more general form than that for free particles and describes a number of practically important cases (see Refs. [11, 16]). Therefore, this confirmation of the validity of the Eriksen method considerably improves the situation with its justification.

The transformation considered above, which is equivalent to taking a square root in the operator sense, can be performed in a general case. If the operators $\mathcal{E}$ and $\mathcal{O}$ do not commute, then, in the weak field approximation ($|\mathcal{E}| \ll m$) and taking into account only unary and binary commutators,

$$\sqrt{\mathcal{H}_D^2} = \epsilon + \frac{1}{4}\left\{\frac{1}{\epsilon}, \{(\beta m + \mathcal{O}), \mathcal{E}\}\right\} - \frac{1}{8}\left\{\frac{\beta m + \mathcal{O}}{\epsilon}, [\epsilon, [\epsilon, \mathcal{E}]]\right\}. \tag{14}$$

Even in this case the subsequent calculations become rather cumbersome, although they can be carried out analytically for certain specific problems. In the case of abandoning the weak field approximation, the expression for $\sqrt{\mathcal{H}_D^2}$ itself becomes rather cumbersome.

One important advantage of step-by-step methods is that they require far fewer calculations to be done. As a consequence, these methods were successfully used to carry out the FW transformation in describing the interaction of relativistic particles with external fields (see Refs. [10, 17] and references therein). Finding a correct and exact expression for the Hamiltonian operator in the present paper by using the Eriksen method and under condition (8) gives good reason to consider that the results derived in [7] are exact ones and, therefore, makes it possible to set the limits of applicability of step-by-step methods. This is done by comparing the relativistic formulas for a Hamiltonian operator in FW representation, which is obtained by using step-by-step methods with the known expression for the first terms of a series defining the expansion of this operator in powers of v/c found in [7] by applying the Eriksen method.

Acknowledgments. This work was supported by the Belarusian Republican Foundation for Fundamental Research, grant No. Φ12D-002.

---

[1] *Foldy L.L., Wouthuysen S.A.* On the Dirac Theory of Spin 1/2 Particles and Its Non-Relativistic Limit // Phys. Rev. 1950. V. 78, No. 1. P. 29-36.

[2] *Costella J.P., McKellar B.H.J.* The Foldy-Wouthuysen transformation // Am. J. Phys. 1995. V. 63, Iss. 12. P. 1119-1121.

[3] *Neznamov V.P., Silenko A.J.* Foldy-Wouthuysen wave functions and conditions of transformation between Dirac and Foldy-Wouthuysen representations // J. Math. Phys. 2009. V. 50, Iss. 12. P. 122302.

[4] *Eriksen E., Korlsrud M.* Canonical transformations of Dirac's equation to even forms. Expansion in terms of the external fields // Nuovo Cimento Suppl. 1960. V. 18, No. 1. P. 1-39.

[5] *Eriksen E.* Foldy-Wouthuysen Transformation. Exact Solution with Generalization to the Two-Particle Problem // Phys. Rev. 1958. V. 111, No. 3. P. 1011-1016.

[6] *Hausdorff F.* Die symbolische Exponentialformel in der Gruppentheorie // Ber. Verh. Saechs. Akad. Wiss. Leipzig, Math.-Phys. Kl. 1906. V. 58. P. 19-48.

[7] *De Vries E., Jonker J.E.* Non-relativistic approximations of the Dirac Hamitonian // Nucl. Phys. B. 1968. V. 6, Iss. 3. P. 213-225.

[8] *Blount E.I.* Extension of the Foldy-Wouthuysen Transformation // Phys. Rev. 1962. V. 128, Iss. 5. P. 2454-2458.

[9] *Silenko A.J.* Dirac equation in the Foldy-Wouthuysen representation describing the interaction of spin-1/2 relativistic particles with an external electromagnetic field // Theor. Math. Phys. 1995. V. 105, No. 1. P. 46-54.

[10] *Silenko A.J.* Foldy-Wouthuysen transformation for relativistic particles in external fields // J. Math. Phys. 2003. V. 44, No. 7. P. 2952-2966.

[11] *Silenko A.J.* Foldy-Wouthyusen transformation and semiclassical limit for relativistic particles in strong external fields // Phys. Rev. A. 2008. V. 77, Iss. 1. P. 012116.

[12] *Gosselin P., Berard A., and Mohrbach H.* Semiclassical diagonalization of quantum Hamiltonian and equations of motion with Berry phase corrections // Eur. Phys. J. B. 2007. V. 58, No. 2. P. 137-148.

[13] *Gosselin P., Berard A., and Mohrbach H.* Semiclassical Dynamics of Dirac Particles Interact-

ing with a Static Gravitational Field // Phys. Lett. A. 2007. V. 368, Iss. 5. P. 356-361.

[14] *Gosselin P., Hanssen J., and Mohrbach H.* Recursive diagonalization of quantum Hamiltonians to all orders in $\hbar$ // Phys. Rev. D. 2008. V. 77, Iss. 8. P. 085008.

[15] *Gosselin P. and Mohrbach H.* Diagonal representation for a generic matrix valued quantum Hamiltonian // Eur. Phys. J. C. 2009. V. 64, No. 3. P. 495-527.

[16] *Silenko A.J. and Teryaev O.V.* Equivalence principle and experimental tests of gravitational spin effects // Phys. Rev. D. 2007. V. 76, Iss. 6. P. 061101.

[17] *Obukhov Yu.N., Silenko A.J. and Teryaev O.V.* Dirac fermions in strong gravitational fields // Phys. Rev. D. 2011. V. 84, Iss. 2. P. 024025.