Scalar particle in general inertial and gravitational fields and conformal invariance revisited

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(Dated: August 7, 2013)

The new manifestation of conformal invariance for a massless scalar particle in a Riemannian spacetime of general relativity is found. Conformal transformations conserve the Hamiltonian and wave function in the Foldy-Wouthuysen representation. Similarity of manifestations of conformal invariance for massless scalar and Dirac particles is proved. New exact Foldy-Wouthuysen Hamiltonians are derived for both massive and massless scalar particles in a general static spacetime and in a frame rotating in the Kerr field approximated by a spatially isotropic metric. The latter case covers an observer on the ground of the Earth or on a satellite and takes into account the Lense-Thirring effect. High-precision formulas are obtained for an arbitrary spacetime metric. General quantum-mechanical equations of motion are derived. Their classical limit coincides with corresponding classical equations.

PACS numbers: 03.65.Pm, 04.20.Jb, 11.10.Ef, 11.30.-j

INTRODUCTION

Penrose [1] has discovered fifty years ago the conformal invariance of the covariant Klein-Gordon (KG) equation [2] for a massless scalar particle in a Riemannian spacetime added by an appropriate term describing a nonminimal coupling to the scalar curvature. Chernikov and Tagirov [3] have given clear explanations of this wonderful result. Their study involved the case of a nonzero mass and n-dimensional Riemannian spacetime. The inclusion of the Penrose-Chernikov-Tagirov term has been argued for both massive and massless particles [3]. The next step in investigation of the problem of conformal invariance of the KG equation has been made by Accioly and Blas [4]. They have performed the exact Foldy-Wouthuysen (FW) transformation for a massive spin-0 particle in static spacetimes and have found new telling arguments in favor of the predicted coupling to the scalar curvature. A derivation of the relativistic FW Hamiltonian is important for a comparison of gravitational (and inertial) effects for scalar and Dirac particles. However, the transformation method used in Ref. [4] is inapplicable to massless particles. In addition, it cannot be applied for nonstatic spacetimes. This does not allow us to obtain information about a manifestation of the conformal invariance in the FW representation.

In the present work, we consider a scalar particle in arbitrary spacetimes in the framework of general relativity (GR). To obtain a Hamiltonian form of the initial covariant KG equation not only for massive particles but also for massless ones, we use the generalization of the Feshbach-Villars transformation [5] proposed in Ref. [6]. Then we fulfill the FW transformation and prove the conformal invariance of the relativistic FW Hamiltonian for a wide class of inertial and gravitational fields. We derive general quantum-mechanical equations of motion and obtain their classical limit.

We denote world and spatial indices by greek and latin letters $\alpha,\mu,\nu,\ldots=0,1,2,3,\ i,j,k,\ldots=1,2,3$, respectively. Tetrad indices are denoted by latin letters from the beginning of the alphabet, $a,b,c,\ldots=0,1,2,3$. Temporal and spatial tetrad indices are distinguished by hats. The signature is $(+---)$, the Ricci scalar curvature is defined by $R=g^{\mu\nu}R_{\mu\nu}=g^{\mu\nu}R^{\alpha}{}_{\mu\alpha\nu}$, where $R^{\alpha}{}_{\mu\beta\nu}=\partial_\beta\Gamma^{\alpha}{}_{\mu\nu}-\ldots$ is the Riemann curvature tensor. We use the system of units $\hbar=1,\ c=1$ except for some specific expressions.

IMPORTANCE OF THE PENROSE-CHERNIKOV-TAGIROV TERM

The covariant KG [2] equation with the additional term [1, 3] describes a scalar particle in a Riemannian spacetime and is given by

$$(\Box+m^2-\lambda R)\psi=0,\quad \Box\equiv\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu. \tag{1}$$

Minimal (zero) coupling corresponds to $\lambda=0$, while the Penrose-Chernikov-Tagirov coupling is defined by $\lambda=1/6$ [7]. For noninertial (accelerated and rotating) frames, the spacetime is flat and $R=0$.

For massless particles, the conformal transformation

$$\widetilde{g}_{\mu\nu}=O^{-2}g_{\mu\nu} \tag{2}$$

conserves the form of Eq. (1) but changes the operators and the wave function [1, 3]:

$$(\widetilde{\Box}-\frac{1}{6}\widetilde{R})\widetilde{\psi}=0,\quad \widetilde{\psi}=O\psi. \tag{3}$$

In Ref. [3], higher dimensionality was also considered.

The corresponding classical equation

$$g^{\mu\nu}p_\mu p_\nu - m^2 = 0$$

is also conformal for a massless particle. It does not contain any nonminimal coupling to the scalar curvature. Therefore, the square of the classical momentum corresponds to the operator $-\hbar^2(\Box - R/6)$ [3].

Chernikov and Tagirov [3] have shown the importance of the additional term for massive particles. They have proved that the requirement for motion to be quasiclassical for a large momentum is satisfied for massive and massless particles only when $\lambda = 1/6$. This choice of λ has been additionally substantiated in Refs. [8, 9].

An important development of problem of the Penrose-Chernikov-Tagirov coupling in the GR for *massive* particles has been made by Accioly and Blas [4]. They analyzed a dependence of the form of the FW Hamiltonian on the value of λ and considered the diagonal static metric

$$ds^2 = V(\boldsymbol{r})^2(dx^0)^2 - W(\boldsymbol{r})^2(d\boldsymbol{r})^2 \tag{4}$$

with arbitrary $V(\boldsymbol{r}), W(\boldsymbol{r})$. The choice of the metric allowed an *exact* FW transformation by the method used in Ref. [4]. This method included the Feshbach-Villars transformation (inappropriate for massless particles) in order to bring the initial equation (1) to the Hamiltonian form. Next, nonunitary and FW transformations resulted in the FW Hamiltonian [4]:

$$\mathcal{H}_{FW} = \rho_3\sqrt{m^2V^2 + \mathcal{F}\boldsymbol{p}^2\mathcal{F} - \frac{1}{4}\nabla\mathcal{F}\cdot\nabla\mathcal{F} + \mathcal{D}_\lambda(V, W)}, \tag{5}$$

where $\boldsymbol{p} = -i\nabla$ is the momentum operator and ρ_i ($i = 1, 2, 3$) are the Pauli matrices. Only for $\lambda = 1/6$, the Darwin term $\mathcal{D}_\lambda(V, W)$ has the simple form and is equal to $\mathcal{F}\Delta\mathcal{F}/6$ [4].

However, the important result obtained by Accioly and Blas [4] demonstrates only a shadow of the conformal invariance, because it does not cover the case of $m = 0$. We perform general examination of the problem.

GENERALIZED FESHBACH-VILLARS TRANSFORMATION

The general form of the covariant KG equation reads

$$\begin{aligned}&\left(\partial_0^2 + \frac{1}{g^{00}\sqrt{-g}}\left\{\partial_i, \sqrt{-g}g^{0i}\right\}\partial_0\right.\\ &\left.+\frac{1}{g^{00}\sqrt{-g}}\partial_i\sqrt{-g}g^{ij}\partial_j + \frac{m^2 - \lambda R}{g^{00}}\right)\psi = 0.\end{aligned} \tag{6}$$

The curly bracket $\{\ldots, \ldots\}$ denotes the anticommutator.

There is an ambiguity [10] in the definition of the parameter of the Feshbach-Villars transformation. We use the generalized Feshbach-Villars transformation proposed in Ref. [6] and based on this ambiguity. In the considered case, the transformation consists in the following definition of components of the wave function:

$$\begin{aligned}&\psi = \phi + \chi, \quad i\left(\partial_0 + \Upsilon\right)\psi = N(\phi - \chi),\\ &\Upsilon = \frac{1}{2g^{00}\sqrt{-g}}\left\{\partial_i, \sqrt{-g}g^{0i}\right\},\end{aligned} \tag{7}$$

where N is an arbitrary nonzero real parameter. For the Feshbach-Villars transformation, it is definite and equal to the particle mass m. This generalization allows us to represent Eq. (6) in the Hamiltonian form describing both massive and massless particles:

$$\begin{aligned}&i\frac{\partial\Psi}{\partial t} = \mathcal{H}\Psi, \quad \mathcal{H} = \rho_3\frac{N^2+T}{2N} + i\rho_2\frac{-N^2+T}{2N} - i\Upsilon,\\ &T = \frac{1}{g^{00}\sqrt{-g}}\partial_i\sqrt{-g}g^{ij}\partial_j + \frac{m^2 - \lambda R}{g^{00}} - \Upsilon^2.\end{aligned} \tag{8}$$

Similarly to Ref. [4], then we perform the nonunitary transformation $\Psi' = f\Psi$ to obtain a pseudo-Hermitian (more exactly, ρ_3-pseudo-Hermitian) Hamiltonian: $\mathcal{H}' = f\mathcal{H}f^{-1}$, $\mathcal{H}' = \rho_3\mathcal{H}'^\dagger\rho_3$. In the case under consideration,

$$\begin{aligned}&f = \sqrt{g^{00}\sqrt{-g}}, \quad \Upsilon' = \frac{1}{2f}\left\{\partial_i, \sqrt{-g}g^{0i}\right\}\frac{1}{f},\\ &T' = \frac{1}{f}\partial_i\sqrt{-g}g^{ij}\partial_j\frac{1}{f} + \frac{m^2 - \lambda R}{g^{00}} - (\Upsilon^2)'.\end{aligned} \tag{9}$$

Transformed operators are denoted by primes and $(\Upsilon^2)' = (\Upsilon')^2$. Tedious but simple calculations result in

$$\begin{aligned}&\mathcal{H}' = \rho_3\frac{N^2 + T'}{2N} + i\rho_2\frac{-N^2 + T'}{2N} - i\Upsilon',\\ &T' = \partial_i\frac{G^{ij}}{g^{00}}\partial_j + \frac{m^2 - \lambda R}{g^{00}} + \frac{1}{f}\nabla_i\left(\sqrt{-g}G^{ij}\right)\nabla_j\left(\frac{1}{f}\right)\\ &+\sqrt{\frac{\sqrt{-g}}{g^{00}}}G^{ij}\nabla_i\nabla_j\left(\frac{1}{f}\right) + \frac{1}{4f^4}\left[\nabla_i(\Gamma^i)\right]^2\\ &-\frac{1}{2f^2}\nabla_i\left(\frac{g^{0i}}{g^{00}}\right)\nabla_j\left(\Gamma^j\right) - \frac{g^{0i}}{2g^{00}f^2}\nabla_i\nabla_j\left(\Gamma^j\right),\\ &\Upsilon' = \frac{1}{2}\left\{\partial_i, \frac{g^{0i}}{g^{00}}\right\}, \quad G^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}, \quad \Gamma^i = \sqrt{-g}g^{0i},\end{aligned} \tag{10}$$

where the nabla operators act only on the operators in brackets. Equation (10) is exact and covers any inertial and gravitational fields.

FOLDY-WOUTHYUSEN TRANSFORMATION

General methods of the FW transformation for relativistic particles have been developed in Refs. [11, 12]. They belong to step-by-step methods performing the transformation as a result of subsequent iterations. We use the version [6] adapted to scalar particles. In this case, the relativistic FW transformation is carried out with the ρ_3-pseudounitary operator ($U^\dagger = \rho_3 U^{-1}\rho_3$) [6]

$$U = \frac{\epsilon + N + \rho_1(\epsilon - N)}{2\sqrt{\epsilon N}}, \quad \epsilon = \sqrt{T'}. \tag{11}$$

It is important that the Hamiltonian obtained as a result of the transformation does not depend on N [6]. This shows a self-consistency of the used transformation method. Next transformation [6] eliminates residual odd terms and leads to the final form of the *approximate* relativistic FW Hamiltonian:

$$\mathcal{H}_{FW} = \rho_3\epsilon - i\Upsilon' - \frac{1}{2\sqrt{\epsilon}}\left[\sqrt{\epsilon}, \left[\sqrt{\epsilon}, (i\partial_0 + i\Upsilon')\right]\right]\frac{1}{\sqrt{\epsilon}}. \tag{12}$$

EXACT FOLDY-WOUTHUYSEN TRANSFORMATION AND CONFORMAL INVARIANCE

The used method ensures the exact FW transformation for a wide class of spacetime metrics. The manifestation of conformal invariance can also be investigated in detail.

The sufficient condition of the exact FW transformation [6, 11, 12] applied to scalar particles is given by $\partial_0 T' - [T', \Upsilon'] = 0$. When it is satisfied, the exact FW Hamiltonian reads

$$\mathcal{H}_{FW} = \rho_3\sqrt{T'} - i\Upsilon'. \tag{13}$$

Equation (13) covers *all static spacetimes* ($\Upsilon' = 0$) and some important cases of stationary ones.

Since general expressions for the scalar Ricci curvature are very cumbersome, we restrict ourselves to an analysis of several special cases. For the metric defined by Eq. (4), the result of our calculations formally coincides with Eq. (5). However, the case of $m = 0$ can now be considered. The explicit expression for $\mathcal{D}_\lambda(V, W)$ [4] shows the presence of conformal invariance for massless particles if and only if $\lambda = 1/6$. In this case, conformal transformation (2) does not change the FW Hamiltonian and the FW wave function Ψ_{FW}. These manifestations of conformal invariance radically differ from those for the covariant KG equation and the corresponding wave function.

The validity of the found properties can be checked for the scalar particle in nonstatic spacetimes. The metric of the rotating Kerr source has been reduced to the Arnowitt-Deser-Misner form [13] by Hergt and Schäfer [14]. This form reproduces the Kerr solution only approximately. The form of the metric can be additionally simplified due to an introduction of spatially isotropic coordinates and dropping terms violating the isotropy [15]:

$$\begin{aligned} ds^2 = V^2(dx^0)^2 - W^2\delta_{ij}(dx^i - K^i dx^0)(dx^j \\ -K^j dx^0), \quad \boldsymbol{K} = \boldsymbol{\omega}\times\boldsymbol{r}. \end{aligned} \tag{14}$$

The use of the approximate Kerr metric allows us to fulfill the *exact* FW transformation when V, W, and $\boldsymbol{\omega}$ depend only on the isotropic radial coordinate r. In this approximation, the metric is defined by

$$\begin{aligned} V(r) = \frac{\kappa_-}{\kappa_+} + \mathcal{O}\left(\frac{\mu a^2}{r^3}\right), \quad W(r) = \kappa_+^2 + \mathcal{O}\left(\frac{\mu a^2}{r^3}\right), \\ \boldsymbol{\omega}(r) = \frac{2\mu c}{r^3}\,\boldsymbol{a}\left[1 - \frac{3\mu}{r} + \frac{21\mu^2}{4r^2} + \mathcal{O}\left(\frac{a^2}{r^2}\right)\right]. \end{aligned} \tag{15}$$

Here $\kappa_\pm = 1 \pm \mu/(2r)$, $\boldsymbol{a} = \boldsymbol{J}/(Mc)$, $\mu = GM/c^2$; the total mass M and the total angular momentum $\boldsymbol{J}$ (directed along the z axis) define the Kerr source uniquely. The leading term in the expression for $\boldsymbol{\omega}(r) = \omega(r)\boldsymbol{e}_z$ corresponds to the Lense-Thirring approximation.

We can pass on from the Kerr field approximated by Eqs. (14) and (15) to a frame rotating in this field with the angular velocity $\boldsymbol{o}$ after the transformation $dx^i \to dX^i = dx^i + (\boldsymbol{o}\times\boldsymbol{r})dx^0$ [16]. The stationary metric of this frame can be obtained from Eqs. (14) and (15) with the replacement $\boldsymbol{\omega} \to \boldsymbol{\Omega} = \boldsymbol{\omega} - \boldsymbol{o}$. In particular, it covers an observer on the ground of a rotating source like the Earth or on a satellite. In this case, $\boldsymbol{o} = \boldsymbol{J}/I$, where I is the moment of inertia. The exact FW Hamiltonian is given by Eq. (13) where

$$\begin{aligned} T' = m^2V^2 + \mathcal{F}\boldsymbol{p}^2\mathcal{F} - \frac{1}{4}\nabla\mathcal{F}\cdot\nabla\mathcal{F} + \mathcal{D}_\lambda(V, W) \\ + \frac{\lambda}{2}(x^2 + y^2)(\Omega_r')^2, \quad \mathcal{D}_\lambda(V, W) = \lambda\mathcal{F}\Delta\mathcal{F} \\ + (1 - 6\lambda)\frac{V}{2W^2}\left[\mathcal{F}\left(\frac{2W_r'}{r} + W_{rr}''\right) + \frac{2V_r'}{r} + V_{rr}''\right], \\ -i\Upsilon' = \boldsymbol{\Omega}\cdot(\boldsymbol{r}\times\boldsymbol{p}), \end{aligned} \tag{16}$$

and derivatives with respect to r are denoted by indices. In particular, for the Lense-Thirring metric $\boldsymbol{\Omega}(r) = 2G\boldsymbol{J}/r^3$, $V(r) = 1 - GM/r$, $W(r) = 1 + GM/r$.

While the metric, (14) and (15), reproduces the Kerr solution only approximately, the derivation of the exact FW Hamiltonian corresponding to this metric allows an independent unambiguous determination of the value of λ. If and only if $\lambda = 1/6$, then the conformal transformation (2) changes neither T' nor $\mathcal{H}_{FW}$, Ψ_{FW}. This property is the same as for the static metric.

GENERAL EQUATIONS OF MOTION

The equations for the FW Hamiltonian allow us to derive general quantum-mechanical equations of motion and then obtain their classical limit ($\hbar \to 0$). The quantum-mechanical equations of motion defining the force, velocity, and acceleration read ($p_0 \equiv \mathcal{H}_{FW}$)

$$\begin{aligned} F^i \equiv \frac{dp^i}{dt} = \frac{1}{2}\frac{\partial}{\partial t}\left\{g^{i\mu}, p_\mu\right\} + \frac{i}{2\hbar}\left[\mathcal{H}_{FW}, \left\{g^{i\mu}, p_\mu\right\}\right], \\ \mathcal{V}^i \equiv \frac{dx^i}{dt} = \frac{i}{\hbar}\left[\mathcal{H}_{FW}, x^i\right], \quad \mathcal{W}^i = \frac{\partial\mathcal{V}^i}{\partial t} + \frac{i}{\hbar}\left[\mathcal{H}_{FW}, \mathcal{V}^i\right]. \end{aligned} \tag{17}$$

Any commutation adds the factor $\hbar$ as compared with the product of operators.

It has been proved in Ref. [17] that satisfying the condition of the Wentzel-Kramers-Brillouin approximation allows us to use this approximation in the relativistic case and to obtain a classical limit of the relativistic quantum mechanics. Determination of the classical limit reduces to the replacement of operators in the FW Hamiltonian and quantum-mechanical equations of motion in the FW representation by the respective classical quantities. The classical limit of the general FW Hamiltonian is given by

$$H=\left(\frac{m^2-G^{ij}p_ip_j}{g^{00}}\right)^{1/2}-\frac{g^{0i}p_i}{g^{00}}. \tag{18}$$

It coincides with the classical Hamiltonian derived in Ref. [18]. The classical limit of Eq. (17) reads

$$\begin{gathered}\mathcal{V}^i=\frac{G^{ij}p_j}{\sqrt{g^{00}(m^2-G^{ij}p_ip_j)}}+\frac{g^{0i}}{g^{00}},\\ F^i=p_\mu\frac{\partial g^{i\mu}}{\partial t}+g^{0i}\frac{\partial H}{\partial t}+g^{ij}\partial_jH+p_\mu\mathcal{V}^j\partial_jg^{i\mu}.\end{gathered} \tag{19}$$

It coincides with the corresponding classical equations which follow from Hamiltonian (18) and the Hamilton equations. Thus, the quantum-mechanical and classical equations are in the best compliance.

For example, the exact metric of a general noninertial frame characterized by the acceleration $\boldsymbol{a}$ and the rotation $\boldsymbol{o}$ of an observer is defined by $V=1+\boldsymbol{a}\cdot\boldsymbol{r}$, $W=1$, $\boldsymbol{\Omega}=-\boldsymbol{o}$ [19]. In this case, the classical limit of the equations of motion is given by ($\boldsymbol{p}\equiv(-p_1,-p_2,-p_3)$)

$$\begin{gathered}\boldsymbol{\mathcal{V}}=(1+\boldsymbol{a}\cdot\boldsymbol{r})\frac{\boldsymbol{p}}{\sqrt{m^2+\boldsymbol{p}^2}}-\boldsymbol{o}\times\boldsymbol{r},\\ \boldsymbol{\mathcal{W}}=-\boldsymbol{a}(1+\boldsymbol{a}\cdot\boldsymbol{r})-2\boldsymbol{o}\times\boldsymbol{\mathcal{V}}-\boldsymbol{o}\times(\boldsymbol{o}\times\boldsymbol{r})\\ +\frac{2\boldsymbol{a}\cdot\boldsymbol{\mathcal{V}}+\boldsymbol{a}\cdot(\boldsymbol{o}\times\boldsymbol{r})}{1+\boldsymbol{a}\cdot\boldsymbol{r}}\left(\boldsymbol{\mathcal{V}}+\boldsymbol{o}\times\boldsymbol{r}\right).\end{gathered} \tag{20}$$

Equation (20) agrees with the classical results [20].

CONFORMAL INVARIANCE FOR DIRAC AND CLASSICAL PARTICLES

It is important to compare the conformal transformations in the GR for massless scalar, Dirac, and classical particles. Our analysis shows that the general Hermitian Dirac Hamiltonian for a massless particle in an arbitrary metric in the presence of an electromagnetic field [15] is not changed by the transformation (2). The FW transformation operator for particles in strong external fields obtained in Ref. [12] is also conformally invariant in the case of $m=0$. As a result, the Dirac and FW wave functions, ψ and ψ_{FW}, and the FW Hamiltonian remain unchanged. These properties of Dirac particles are the same as for scalar ones. The Hamiltonian of massless classical particles is conformally invariant even if its spin-dependent part defined by Eqs. (3.18) and (4.12) in Ref. [15] is taken into account.

In the general case, the transformation of the initial covariant Dirac equation to the Hermitian Hamiltonian form is performed by the nonunitary operator $f_D=(\sqrt{-g}e^0_{\widehat{0}})^{1/2}$ [15]. Since the transformation (2) leads to $\widetilde{f}_D=O^{-3/2}f_D$, the conformally transformed wave function of the initial covariant Dirac equation, $\widetilde{\Psi}$, reads

$$\widetilde{\Psi}=\widetilde{f_D^{-1}}\widetilde{\psi}=O^{3/2}f_D^{-1}\psi=O^{3/2}\Psi. \tag{21}$$

While its transformation is similar to that for the scalar particles, the powers of O in Eqs. (3) and (21) differ.

The second-order wave equation for the Dirac particles in general electromagnetic and gravitational fields derived in Ref. [15] includes the term describing a nonminimal coupling to the scalar curvature R. As the definitions of R in Ref. [15] and the present work differ in sign, this term corresponds to $\lambda=1/4$.

CONCLUSIONS

The use of the generalized Feshbach-Villars and relativistic FW transformations allows us to describe the both massive and massless scalar particles in general noninertial frames and gravitational fields. The present work demonstrates the new manifestation of the conformal invariance for massless particles. The conformal transformation conserves the FW Hamiltonian and the FW wave function while it changes the wave function of the initial KG equation. The similar conclusion is valid for the Dirac particles. The nonminimal coupling to the scalar curvature is not a unique property of scalar particles.

The results obtained in Ref. [4] and in the present study allow us to state the general property of conformal symmetry for massive particles. Conformal transformation (2) changes only such terms in the FW Hamiltonian which are proportional to the particle mass m. This property is valid not only for real scalars (Higgs boson) but also for compound ones (zero-spin atoms and nuclei).

Contemporary methods of (pseudo)unitary and nonunitary transformations make it possible to derive new exact FW Hamiltonians for both massive and massless scalar particles (i) in the general static spacetime and (ii) in the frame rotating in the Kerr field approximated by a spatially isotropic metric. The latter result covers an observer on the ground of the Earth or on a satellite. It reproduces not only the well-known effects of the rotating frame but also the Lense-Thirring effect. For an arbitrary metric, high-precision formula (12) is obtained. The classical limit of the derived general quantum-mechanical equations of motion coincides with corresponding classical equations.

Acknowledgements

The author is indebted to E. A. Tagirov for his interest in the present study and valuable discussions and Yu. A. Tsalkou for the analytic computer calculation of R. The work was supported by the Belarusian Republican Foundation for Fundamental Research (Grant No. Φ12D-002).

[1] R. Penrose, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt, (Gordon and Breach, London, 1964), p. 565.

[2] O. Klein, Z. Phys. **37**, 895 (1926); W. Gordon, Z. Phys. **40**, 117 (1926). The equation has been first obtained by E. Schroedinger (unpublished) and also by V. Fock, Z. Phys. **38**, 242 (1926).

[3] N. Chernikov and E. Tagirov, Ann. Inst. Henri Poincaré A **9**, 109 (1968).

[4] A. Accioly and H. Blas, Phys. Rev. D **66**, 067501 (2002); Mod. Phys. Lett. A **18**, 867 (2003).

[5] H. Feshbach and F. Villars, Rev. Mod. Phys. **30**, 24 (1958).

[6] A.J. Silenko, Teor. Mat. Fiz. **156**, 398 (2008) [Theor. Math. Phys. **156**, 1308 (2008)].

[7] The sign of the Penrose-Chernikov-Tagirov term depends on the definition of R.

[8] S. Sonego and V. Faraoni, Classical Quantum Gravity **10**, 1185 (1993); V. Faraoni, Phys. Rev. D **53**, 6813 (1996).

[9] A. Grib and E. Poberii, Helv. Phys. Acta **68**, 380 (1995).

[10] A. Mostafazadeh, J. Phys. A **31**, 7829 (1998); Ann. Phys. **309**, 1 (2004).

[11] A. J. Silenko, J. Math. Phys. **44**, 2952 (2003).

[12] A. J. Silenko, Phys. Rev. A **77**, 012116 (2008).

[13] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[14] S. Hergt and G. Schäfer, Phys. Rev. D **77**, 104001 (2008).

[15] Yu. N. Obukhov, A. J. Silenko, and O. V. Teryaev, Phys. Rev. D **84**, 024025 (2011).

[16] Frames rotating in the isotropic and Cartesian coordinates are not equivalent.

[17] A. J. Silenko, Pis'ma Zh. Fiz. Elem. Chast. Atom. Yadra **10**, 144 (2013) [Phys. Part. Nucl. Lett. **10**, 91 (2013)].

[18] G. Cognola, L. Vanzo, and S. Zerbini, Gen. Relativ. Gravit. **18**, 971 (1986).

[19] F. W. Hehl and W. T. Ni, Phys. Rev. D **42**, 2045 (1990).

[20] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 175; H. Goldstein, C. P. Poole, and J. L. Safko, *Classical Mechanics* (Addison-Wesley, San Francisco, 2001), 3rd ed., p. 175.